SOME PROBLEMS RELATED TO QUEUEING AND NETWORKS

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BY

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DECLARATION

This thesis entitled "**Some Problems Related to Queueing and Networks.**" submitted in Department of Mathematics and Statistics, Bundelkhand University, Jhansi (U.P.), by me, for the award of the degree of Doctor of Philosophy is based on my research work carried on under the supervision of **Dr. V.K.Sehgal.**

The work, either in part or in full has not been submitted to any university or institution for the award of any degree.

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CHAPTER ONE

1.1 INTRODUCTION

The Queueing networks is a current area of great research and application intersect with many extremely difficult problems. It increased applicability to modeling computer and communication nets. Disney (1981), Kelley (1979) and Lemoine (1977) referred it further.

Kelley (1975) has studied the behaviour of equilibrium of networks of queues in which customers may be different types. The type of a customer is allowed to influence his choice of path through the network and under certain conditions his service time distribution at each queue. The model assumed will usually cause each service time distribution to be of a form related to the negative exponential distribution. Kelley (1976) has obtained equilibrium distribution and in certain cases it is shown that the state of an individual queue is independent of the state of the rest of the network.

Networks of queues described as a group of nodes where each node represents a service facility of some kind. The customer may arrive from outside the system to any node and may depart from the system from any node.. Customers

may returned to nodes previously visited, skip some nodes entirely, and even choose to remain in the system forever. Arrivals from the outside to node follow a poisson process and service times for each channel at node are independent and exponentially distributed. Networks that have these properties are called Jackson networks (1957, 1963).

When no customer may enter the system from the outside and no customer may leave the system are known as closed Jackson networks. When the customers flow in a circle always from node 1 to node 2 and then back to node 1, such closed network system is known as cyclic queues. For open network when customer may enter from outside only at node 1 and depart from node K, we generalize the series to a true open network. These networks are called series queues. In series queues the initial input rate is poisson, the service at all stations is exponential and there is no restriction on queue size between stations. When there are limits on the capacity of a station, this form the blocking effect, that is, a station down stream comes up to capacity and thereby prevents any further processing at upstream stationswhich feed it.

A single server at each station model where no queue is allowed to form at either station is known as simple

sequential two station. If a customer is in station two and service completed at station one, the station one customer must wait there until the station two customer is completed, that is, the system is blocked. Arrivals at station one when the system is blocked are turned away. If a customer is in the process at the station one, even if the station two is empty, arriving customers are turned away, since the system is a sequential one, that is all customers require service at one and then service at two. The problem expands if one allow limits other than zero on queue length or considers more stations. If one customer is allowed to wait between stations the result is seven state probabilities for which to solve, utilizing seven equations and a boundary condition. The complexity results from having to write a difference equation for each possible system state. These types of series queuing situations can be attached via themethodology. For large numbers of equations, as long as we have a finite set numerical techniques for solving these simultaneous equations can also be employed.

Hunt (1956) treated a modified series model using finite difference operators to solve a two station sequential series queue in which no waiting is allowed between stations, but where a queue with no limit is

permitted in-front of the first station. He obtained the steady state probabilities for this model, the expected system size and the maximum allowable for steady state to be assured. He also calculated the maximum allowable for some generalizations of this two station model to three and four station systems with no waiting between stations.

In open network customer can arrive from outside to any node according to a poisson process. At servers at node work according to an exponential distribution. When a customer completes service at node, he goes next to node. Since there is Markovian system, we use our usual types of analysis to write a steady state system equations. Since various numbers of customers can be at various nodes in the network, we desire the joint probability distribution for the number of customers at each node. From this we can obtain the marginal distribution for number of customers a particular node. We use the method of stochastic balance to obtain the steady state equations for this network.

Disney (1981) shows that the actual internal flow in these kinds of network is not poisson. There is any kind of feedback, that is customer can return to previously visited nodes, the internal flows are not poisson. The complexity and intrigue of network waiting time is known as

Sojourn time. Burke (1961) showed that in a three stations series queue with the first and third stations having a single server but the middle station having multiple servers. Simon and Foley (1979) considered a three station queueing network with one server at the first and third stations and multiple servers at the second station.

If there are multiple servers at station other than the first or last so that customer can bypass one another, system sojourn times for successive customers are independent. Malamed (1979) showed for nodes from which units could leave the network, that these departure processes are poisson and that the collections over all nodes that yield these poisson departure processes are mutually independent.

The nodes with no feed back, the output process from thus node is also poisson. In nodes with feedback, one can think of two departing streams one with customers who will either directly or eventually feed back, and other with customers who will not. As long as there is no feed back, as in series or arborescent network, flows between nodes and to the outside are truly poisson feed back destroys poisson flows.

In the closed Jackson network, no customer may enter the system from the outside and no customer may leave

the system. Gordon and Newell (1967) find the product from solution for this network. Buzen (1973) present most useful result for closed Jackson networks. Bruell and Balbo (1980) gave a computational algorithms for closed networks.

A cyclic queue is a sort of series queue in a circle, where the output of the last node feed back to the first node. This is special case of a closed queueing network. Jackson networks have been extended in several ways. First in 1963, for open networks he allowed state-dependent exogenous arrival processes and state dependent internal service. The poisson arrival processes could depend on the total number of units in the network, while the exponential service time could depend on the number of customers present at that node. Computation of the normalizing-constants must be done similarly to that for closed network. Another avenue of generalization of Jackson network is to include travel time always be modeled as another node, but most often these are ample server nodes.

Posner and Bernholtz (1968) treated closed Jackson networks but allowed for ample service travel time nodes with general travel time distribution. For any nodes in a Jackson network with ample service, the forms of service time distributions do not explicitly enter as long as the marginal

distributions of interest do not include these nodes. The final extension of Jackson networks which allow for different classes of customers. A multi class Jackson network is a Jackson network with multiple classes of customers, where each of customers has its own mean arrival rate, its own routing structure and where the mean service times at a node may depend on the particular customer type.

Baskett et al (1975) treated multi-class Jackson networks and obtained product from solutions for processor-sharing, ample service and LCFS with preemptive servicing. They allow the network to be open for some classes of customers and closed for others. Customers may switch classes after finishing at a node according to the probability distribution for 1 server FCFS nodes, service time for all classes are independent and identical distributed exponential.

Kelly's work (1975, 1976, 1979) represents the state of art in the generalization of Jackson network. He also considered multiple customer classes. He further conjectured that many of his results can be extended to include general service time distribution. The conjecture was based on the fact that non negative probability distribution can always be well approximated by finite mixtures of gamma

distributions. Kelly's conjecture is proved by Barbour (1976) Gross and Ince (1981) have applied Kelly's multi class results to a closed networks and obtained numerical solutions for an application in repairable item inventory control. A great deal of effort has been expended in obtaining computational results for closed multi class Jackson networks due to their use in modeling computer system. The basic mode generally consider a computer system with N terminals, one for each user logged on. Since user log on and off during busy period one can assume all terminals are in use so that there are always N customers in the system. These can be at various stages in the system such as "thinking". at the terminal waiting in the queue to enter the Central Processing Unit (CPU), being serviced by the CPU, waiting or in service at input/output stations and so on. Bruell and Balbo (1980) provided a compendium of algorithms developed to treat such models. When the probability that a customer who has completed service at node will go next to node are allowed to be state dependent, then this network is known as non-Jackson network.

1.2 SERIES QUEUES (QUEUE OUTPUT)-

In such type of queue there are a series of service stations through which each calling unit must progress prior to leaving the system.

We assume that the calling unit arrive according to a Poisson process, mean λ and the service time of each server at station i is exponential with mean $1/\mu_i$.

We consider an $M/M/c/\infty$ queue in steady state. Let $N(t)$ now represent the number of customers in the system at a time t after the last departure.

$$Pr \{ N(t) = n \} = p_n$$

Let T represent the random variable "time between successive departures" and

$$F_n(t) = Pr \{N(t) = n \text{ and } T>t\}$$

So $F_n(t)$ is the joint probability that there are n customers in the system at a time t after the last departure. The commutative distribution of the random variable T is given by

$$C(t) = Pr \{T \le t\}$$

$$= 1 - \sum_{n=0}^{\infty} F_n(t)$$

Since $\sum_{n=0}^{\infty} F_n(t) = Pr \{T>t\}$ is the marginal complementary commutative distribution of T.

The difference equation concerning $F_n(t)$.

<u>For $c \leq n$</u>

$\lambda_n = \lambda$ and $\mu_n = c\mu$

$$F_n(t+\Delta t) = F_n(t)\,(1-\lambda_n\Delta t+O(\Delta t))\,(1-\mu_n\Delta t+O(\Delta t))$$

$$+ F_{n-1}(t)\,(\lambda_n\Delta t+O(\Delta t))\,(1-\mu_n\Delta t+O(\Delta t)) + O(\Delta t)$$

$$F_n(t+\Delta t) = F_n(t)\,(1-\lambda\Delta t+O(\Delta t))\,(1-c\mu\Delta t+O(\Delta t))$$

$$+ F_{n-1}(t)\,(\lambda\Delta t+O(\Delta t))\,(1-c\mu\Delta t+O(\Delta t)) + O(\Delta t)$$

Combining all terms of $O(\Delta t)$ and neglecting terms of order $[O(\Delta t)]^2$ and higher, we get

$$F_n(t+\Delta t) - F_n(t) = -(\lambda+c\mu)\Delta t\; F_n(t) + \lambda F_{n-1}(t)\Delta t + O(\Delta t)$$

Divided by Δt and taking limit $\Delta t \longrightarrow 0$

$$\frac{d}{dt} F_n(t) = -(\lambda+c\mu)\; F_n(t) + \lambda F_{n-1}(t)$$

$$\frac{d}{dt} F_n(t) + (\lambda+c\mu)\; F_n(t) = \lambda F_{n-1}(t) \longrightarrow \quad (1.1)$$

<u>For $1 \leq n \leq c$</u>

$\lambda_n = \lambda$ and $\mu_n = n\mu$

$$F_n(t+\Delta t) = F_n(t)\,(1-\lambda_n\Delta t+O(\Delta t))\,(1-\mu_n\Delta t+O(\Delta t))$$

$$+ F_{n-1}(t)\,(\lambda_n\Delta t+O(\Delta t))\,(1-\mu_n\Delta t+O(\Delta t)) + O(\Delta t)$$

$$F_n(t+\Delta t) = F_n(t)\,(1-\lambda\Delta t+O(\Delta t))\,(1-n\mu\Delta t+O(\Delta t))$$

$$+ F_{n-1}(t)\,(\lambda\Delta t+O(\Delta t))\,(1-n\mu\Delta t+O(\Delta t)) + O(\Delta t)$$

Combining all the terms of $O(\Delta t)$ and neglecting terms of order $[O(\Delta t)]^2$ and higher, we get

$$F_n(t+\Delta t) - F_n(t) = -(\lambda+n\mu)(\Delta t)F_n(t) + \lambda F_{n-1}(t)\Delta t + O(\Delta t)$$

Divided by Δt and taking limit as $\Delta t \longrightarrow 0$

$$\frac{d}{dt} P_n(t) = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t)$$

$$\frac{d}{dt} P_n(t) + (\lambda + n\mu) P_n(t) = \lambda P_{n-1}(t) \longrightarrow (1.2)$$

<u>For n=0</u>

$$P_0(t+\Delta t) = P_0(t)\,(1-\lambda\Delta t+o(\Delta t)) + o(\Delta t)$$

$$P_0(t+\Delta t) - P_0(t) = -\lambda\Delta t P_0(t) + o(\Delta t)$$

Dividing by Δt and taking limit as $\Delta t \longrightarrow 0$

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t)$$

$$\frac{d}{dt} P_0(t) + \lambda P_0(t) = 0 \longrightarrow (1.3)$$

Equation (1.3) can be written as

$$\frac{dP_0(t)}{P_0(t)} = -\lambda\Delta t$$

On integrating

$$\log P_0(t) = C - \lambda\Delta t$$

$$P_0(t) = Ce^{-\lambda t}$$

Using boundary conditions $P_0(t) = Pr\{N(0)=0\} = p_0$

at $t=0$ $\qquad P_0(0) = p_0$

$\Rightarrow C = p_0$

$$\therefore \boxed{P_0(t) = p_0 e^{-\lambda t}} \longrightarrow (1.4)$$

From equation (1.1), put n=1 in equation (1.2)

$$\frac{d}{dt} P_1(t) + (\lambda+\mu) P_1(t) = \lambda P_0(t) = \lambda p_0 e^{-\lambda t}$$

$$\therefore IF = \exp \int (\lambda+\mu)\,dt = e^{(\lambda+\mu)t}$$

Solution is

$$P_1(t)\, e^{(\lambda+c\mu)t} = \int e^{(\lambda+c\mu)t} \lambda p_0 e^{-\lambda t}\, dt + C'$$

$$P_1(t)\, e^{(\lambda+c\mu)t} = \lambda p_0 \int e^{c\mu t}\, dt + C'$$

$$P_1(t)\, e^{(\lambda+c\mu)t} = \frac{\lambda}{c\mu} p_0 e^{c\mu t} + C'$$

at $t=0$, $P_1(0) = p_1$

$$\therefore p_1 = \frac{\lambda}{c\mu} p_0 + C'$$

But for $p_1 = \frac{\lambda}{c\mu} p_0 \qquad \Rightarrow C' = 0$

Therefore

$$P_1(t) e^{(\lambda+c\mu)t} = \frac{\lambda}{c\mu} p_0 e^{c\mu t}$$

$$P_1(t) = \frac{\lambda}{c\mu} p_0 e^{-\lambda t}$$

$$\boxed{P_1(t) = p_1 e^{-\lambda t}} \longrightarrow (1.5)$$

Where $p_1 = \frac{\lambda}{c\mu} p_0$

Put n=2 in Equation (1.1)

$$\frac{d}{dt} P_2(t) + (\lambda+c\mu)\, P_2(t) = \lambda P_1(t)$$

$$\frac{d}{dt} P_2(t) + (\lambda+c\mu)\, P_2(t) = \lambda p_1 e^{-\lambda t}$$

$$IF = \exp \int (\lambda+c\mu)\,dt = e^{(\lambda+c\mu)t}$$

Therefore solution become

$$P_2(t) e^{(\lambda+c\mu)t} = \int e^{(\lambda+c\mu)t} \lambda p_1 e^{-\lambda t}\, dt + C'$$

$$P_2(t) e^{(\lambda+c\mu)t} = \lambda p_1 \int e^{c\mu t}\, dt + C'$$

$$P_2(t)e^{(\lambda+c\mu)t} = \frac{\lambda}{c\mu} P_1 e^{\mu c t} + C''$$

At t=0 $P_2(0) = P_2$

Therefore $P_2(0) \cdot 1 = \frac{\lambda}{c\mu} P_1 + C''$

$$P_2 = \frac{\lambda}{c\mu} P_1 + C''$$

But for $P_2 = \frac{\lambda}{c\mu} P_1 \Rightarrow C'' = 0$

Hence $P_2(t)e^{(\lambda+c\mu)t} = \left(\frac{\lambda}{c\mu}\right) P_1 e^{-\lambda t}$

$$P_2(t) = \frac{\lambda}{c\mu} P_1 e^{-\lambda t}$$

$$\boxed{P_2(t) = P_2 e^{-\lambda t}} \longrightarrow (1.6)$$

where $P_2 = \frac{\lambda}{c\mu} P_1$

and so on

In general

$$\left.\begin{aligned} &P_n(t) = P_n e^{-\lambda t} \\ \text{where} \quad &P_{n+1} = \frac{\lambda}{c\mu} P_n \end{aligned}\right] \quad (1.7) \quad n \leq c$$

From Equation (1.2)

Put n=1 in Eq. (1.2)

$$\frac{d}{dt} P_1(t) + (\lambda+\mu)P_1(t) = \lambda P_0(t)$$

$$\frac{d}{dt} P_1(t) + (\lambda+\mu)P_1(t) = \lambda P_0 e^{-\lambda t}$$

$$IF = \exp \int (\lambda+\mu)dt = e^{(\lambda+\mu)t}$$

Therefore

$$P_1(t)e^{(\lambda+\mu)t} = \int e^{(\lambda+\mu)t}\,\lambda p_0 e^{-\lambda t}\,dt + C_1$$

$$P_1(t)e^{(\lambda+\mu)t} = \lambda p_0 \int e^{\mu t} + C_1$$

$$P_1(t)e^{(\lambda+\mu)t} = \frac{\lambda}{\mu}\, p_0 e^{\mu t} + C_1$$

At t=0 $P_1(0) = p_1$

$$p_1 = \frac{\lambda}{\mu}\, p_0 + C_1$$

Or $\qquad p_1 = \frac{\lambda}{\mu}\, p_0 \Rightarrow \boxed{C_1 = 0}$

Hence $P_1(t)\, e^{(\lambda+\mu)t} = \frac{\lambda}{\mu}\, p_0 e^{\mu t}$

$$P_1(t) = \frac{\lambda}{\mu}\, p_0 e^{-\lambda t}$$

$$\boxed{P_1(t) = p_1 e^{-\lambda t}} \longrightarrow (1.7)$$

where $p_1 = \frac{\lambda}{\mu}\, p_0$

Put n=2 in equation (1.2)

$$\frac{d}{dt} P_2(t) + (\lambda+2\mu)P_2(t) = \lambda P_1(t)$$

$$\frac{d}{dt} P_2(t) + (\lambda+2\mu)P_2(t) = \lambda p_1 e^{-\lambda t}$$

$$IF = \exp \int (\lambda+2\mu)\,dt = e^{(\lambda+2\mu)t}$$

$$P_2(t)e^{(\lambda+2\mu)t} = \int e^{(\lambda+2\mu)t}\,\lambda p_1 e^{-\lambda t}\,dt + C_2$$

$$P_2(t)e^{(\lambda+2\mu)t} = \lambda p_1 \int e^{2\mu t}\,dt + C_2$$

$$P_2(t)e^{(\lambda+2\mu)t} = \frac{\lambda}{2\mu}\, p_1 e^{2\mu t} + C_2$$

At $t=0$ $\quad P_2(0) = P_2$

$$\therefore P_2 = \frac{\lambda}{2\mu} P_1 + C_2$$

for $\quad P_2 = \frac{\lambda}{2\mu} P_1 \Rightarrow C_2 = 0$

Hence $\quad P_2(t)e^{(\lambda+2\mu)t} = \frac{\lambda}{2\mu} P_1 e^{2\mu t}$

$$P_2(t) = \frac{\lambda}{2\mu} P_1 e^{-\lambda t}$$

$$\boxed{P_2(t) = P_2 e^{-\lambda t}} \longrightarrow (1.8)$$

where $\quad P_2 = \frac{\lambda}{2\mu} P_1$

and so on

In general for $1 \le n \le c$

$$P_n(t) = P_n e^{-\lambda t}$$

where $\quad P_{n+1} = \frac{\lambda}{(n+1)\mu} P_n$

Hence the solution of equations

$$P_n = P_n e^{-\lambda t}$$

where

$$P_{n+1} = \frac{\lambda}{c\mu} P_n \quad \text{when} \quad c \le n$$

$$P_{n+1} = \frac{\lambda}{(n+1)\mu} P_n \quad \text{when} \quad 1 \le n \le c \qquad \longrightarrow (1.9)$$

1.3 SERIES QUEUES WITH BLOCKING :

We suppose that there are two simple sequential stations, single server at each station model where no queue

is allowed to form at either station. If a customer is in station two, and service is completed at station one, the station one customers must wait there until the station two customer is completed, the system is blocked.

Arrivals at station one when the system is blocked are turned away. Also if a customer is in process at station one, even if station two is empty, arriving customers are turned away, since the system is a sequential one, that is all customers require service at one and then service at two.

To find the steady-state probability $p_{n1,n2}$ of n_1 in the first station and n_2 in the second station. For this model the possible states are given below in Table.

n_1,n_2	Description
0,0	System empty
1,0	Customers in process 1 only
0,1	Customers in process 2 only
1,1	Customers in process 1 and 2
b,1	Customers in process 2 and system 1 is blocked.

Assuming arrivals to the system are poisson with parameter λ and service is exponential with parameters μ_1

and μ_2 respectively.

Hence difference equations written as :

$$P_{0,0}(t+\Delta t) = P_{0,0}(t)\,(1-\lambda\Delta t+O(\Delta t))$$
$$+ P_{0,1}(t)\,(1-\lambda\Delta t+O(\Delta t))\,(\mu_2\Delta t+O(\Delta t))$$

$$P_{0,0}(t+\Delta t) - P_{0,0}(t) = -\lambda P_{0,0}(t)\Delta t + \mu_2 P_{0,1}(t)\Delta t$$
$$+ O(\Delta t) \text{ terms and higher terms}$$

Divided by Δt and taking $\Delta t \longrightarrow 0$

$$\boxed{\frac{d}{dt}P_{0,0}(t) = -\lambda P_{0,0}(t) + \mu_2 P_{0,1}(t)} \longrightarrow (1.10)$$

$$P_{1,0}(t+\Delta t) = P_{1,0}(t)\,(1-\mu_1\Delta t+O(\Delta t))$$
$$+ P_{1,1}(t)\,(1-\mu_1\Delta t+O(\Delta t))\,(\mu_2\Delta t+O(\Delta t))$$
$$+ P_{0,0}(t)\,(\lambda\Delta t+O(\Delta t))$$

$$P_{1,0}(t+\Delta t) - P_{1,0}(t) = -\mu_1 P_{1,0}(t)\Delta t + \mu_2 P_{1,1}(t)\Delta t$$
$$+ \lambda P_{0,0}(t)\Delta t + O(\Delta t) \text{ terms and higher}$$

Divided by Δt and taking $\Delta t \longrightarrow 0$

$$\boxed{\frac{d}{dt}P_{1,0}(t) = -\mu_1 P_{1,0}(t) + \mu_2 P_{1,1}(t) + \lambda P_{0,0}(t)} \longrightarrow (1.11)$$

$$P_{0,1}(t+\Delta t) = P_{0,1}(t)\,(1-\lambda\Delta t+O(\Delta t))\,(1-\mu_2\Delta t+O(\Delta t))$$
$$+ P_{1,0}(t)\,(\mu_1\Delta t+O(\Delta t)) + P_{b,1}(t)\,(\mu_2\Delta t+O(\Delta t))$$

$$P_{0,1}(t+\Delta t) - P_{0,1}(t) = -\lambda P_{0,1}(t)\Delta t - \mu_2 P_{0,1}(t)\Delta t$$
$$+ \mu_1 P_{1,0}(t)\Delta t + \mu_2 P_{b,1}(t)\Delta t + O(\Delta t)) \text{ terms and higher}$$

Divided by Δt and taking $\Delta t \longrightarrow 0$

$$\frac{d}{dt} p_{0,1}(t) = -\lambda p_{0,1}(t) - \mu_2 p_{0,1}(t) + \mu_1 p_{1,0}(t) + \mu_2 p_{b,1}(t)$$

$$\boxed{\frac{d}{dt} p_{0,1}(t) = -(\lambda+\mu_2) p_{0,1}(t) + \mu_1 p_{1,0}(t) + \mu_2 p_{b,1}(t)} \longrightarrow (1.12)$$

$$p_{1,1}(t+\Delta t) = p_{1,1}(t)\,(1-\mu_1\Delta t+O(\Delta t))\,(1-\mu_2\Delta t+O(\Delta t))$$
$$+\, p_{0,1}(t)\,(\lambda\Delta t+O(\Delta t))\,(1-\mu_2\Delta t+O(\Delta t))$$

$$p_{1,1}(t+\Delta t) = p_{1,1}(t) - (\mu_1+\mu_2) p_{1,1}(t)\Delta t + \lambda p_{0,1}(t)\Delta t$$
$$+\, O(\Delta t) \text{ terms and higher}$$

Divided by Δt and taking $\Delta t \longrightarrow 0$

$$\boxed{\frac{d}{dt} p_{1,1}(t) = -(\mu_1+\mu_2) p_{1,1}(t) + \lambda p_{0,1}(t)} \longrightarrow (1.13)$$

$$p_{b,1}(t+\Delta t) = p_{b,1}(t)\,(1-\mu_2\Delta t+O(\Delta t))$$
$$+\, p_{1,1}(t)\,(\mu_1\Delta t+O(\Delta t))\,(1-\mu_2\Delta t+O(\Delta t))$$

$$p_{b,1}(t+\Delta t) = p_{b,1}(t) - \mu_2 p_{b,1}(t)\Delta t + \mu_1 p_{1,1}(t)\Delta t$$
$$+\, O(\Delta t) \text{ terms and higher}$$

Divided by Δt and taking limit $\Delta t \longrightarrow 0$

$$\boxed{\frac{d}{dt} p_{b,1}(t) = -\mu_2 p_{b,1}(t) + \mu_1 p_{1,1}(t)} \longrightarrow (1.14)$$

For steady state solution, we have as $t \longrightarrow \infty$, $p_{n1,n2}(t) \longrightarrow p_{n1,n2}$ and $\frac{d}{dt} p_{n1,n2}(t) \longrightarrow 0$, using these conditions in Equation (1.10) to Equation (1.14); Therefore

$$-\lambda P_{0,0} + \mu_2 P_{0,1} = 0 \longrightarrow (1.15)$$

$$-\mu_1 P_{1,0} + \mu_2 P_{1,1} + \lambda P_{0,0} = 0 \longrightarrow (1.16)$$

$$-(\lambda+\mu_2) P_{0,1} + \mu_1 P_{1,0} + \mu_2 P_{b,1} = 0 \longrightarrow (1.17)$$

$$-(\mu_1+\mu_2) P_{1,1} + \lambda P_{0,1} = 0 \longrightarrow (1.18)$$

$$-\mu_2 P_{b,1} + \mu_1 P_{1,1} = 0 \longrightarrow (1.19)$$

If we suppose that $\mu_1=\mu_2=\mu$, then the results are

from (1.15) $$P_{0,1} = \frac{\lambda}{\mu} P_{0,0} \longrightarrow (1.20)$$

from (1.18) and (1.20) $$2\mu P_{1,1} = \lambda\frac{\lambda}{\mu} P_{0,0}$$

$$P_{1,1} = \frac{\lambda^2}{2\mu^2} P_{0,0} \longrightarrow (1.21)$$

from Eq.(1.19) & Eq.(1.21)

$$\mu P_{b,1} = \mu P_{1,1}$$

$$P_{b,1} = \frac{\lambda^2}{2\mu^2} P_{0,0} \longrightarrow (1.22)$$

from Eq.(1.16) & Eq.(1.21)

$$-\mu P_{1,0} + \mu\frac{\lambda^2}{2\mu^2} P_{0,0} + \lambda P_{0,0} = 0$$

$$\mu P_{1,0} = \left(\lambda + \frac{\lambda^2}{2\mu}\right) P_{0,0}$$

$$P_{1,0} = \frac{\lambda(\lambda+2\mu)}{2\mu^2} P_{0,0} \longrightarrow (1.23)$$

Now using boundary condition

$$\Sigma\,\Sigma\, P_{n1,n2} = 1$$

$$P_{0,0} + P_{1,1} + P_{0,1} + P_{1,0} + P_{1,1} = 1$$

$$P_{0,0} + \frac{\lambda^2}{2\mu^2} P_{0,0} + \frac{\lambda}{\mu} P_{0,0} + \frac{\lambda(\lambda + 2\mu)}{2\mu^2} P_{0,0} + \frac{\lambda^2}{2\mu^2} P_{0,0} = 1 \cdot$$

$$P_{0,0} \left[1 + \frac{\lambda^2}{2\mu^2} + \frac{\lambda}{\mu} + \frac{\lambda^2 + 2\lambda\mu}{2\mu^2} + \frac{\lambda^2}{2\mu^2} \right] = 1$$

$$P_{0,0} \left[\frac{2\mu^2 + \lambda^2 + 2\lambda\mu + \lambda^2 + 2\lambda\mu + \lambda^2}{2\mu^2} \right] = 1$$

$$P_{0,0} \frac{(3\lambda^2 + 4\mu\lambda + 2\mu^2)}{2\mu^2} = 1$$

$$P_{0,0} = \frac{2\mu^2}{(3\lambda^2 + 4\mu\lambda + 2\mu^2)} \longrightarrow (1.24)$$

Hence results are

$$\left[\begin{array}{l} P_{1,0} = \dfrac{\lambda(\lambda + 2\mu)}{2\mu^2} P_{0,0} \\[2em] P_{0,1} = \dfrac{\lambda}{\mu} P_{0,0} \\[2em] P_{1,1} = \dfrac{\lambda^2}{2\mu^2} P_{0,0} \\[2em] P_{1,1} = \dfrac{\lambda^2}{2\mu^2} P_{0,0} \\[2em] P_{0,0} = \dfrac{2\mu^2}{(3\lambda^2 + 4\mu\lambda + 2\mu^2)} \end{array} \right]$$

1.4 OPEN JACKSON NETWORKS

We consider a networks of k service facilities (nodes). Customers can arrive from "outside" to any node according to a poisson process. We will represent the mean arrival rate to node i as γ_i. All servers at node i work according to an exponential distribution with mean μ_i (all servers at a given node are identical).

When a customer complete service at node i, he goes next to node j with probability r_{ij} (i=1,2,..k). There is a probability r_{i0} that a customer will leave the network at node i upon completion of service. There is no limit on queue capacity at any node; that is, we never have a bloked system or node.

Since we have a Markovian system, we can use our usual types of analysis to write the steady state system equations. We first, however, must determine how to describe a system state. Since various numbers of customers can be at various nodes in the networks, we desire the joint probability distribution for the number of customers at each node, that is letting N_i be the random variable for the number of customers at node i in the steady state. We desire

$$Pr\{N_1=n_1, N_2=n_2, \ldots, N_k=n_k\} = p_{(n_1,n_2,\ldots,n_k)}$$

From this joint probability distribution, we can obtain the marginal distribution for the number of customers at a particular node by appropriate summing over the other nodes.

Simplified State Descriptors

State	Simplified Notation
$n_1, n_2, \ldots, n_i, n_j, \ldots, n_k$	$\bar{n}$
$n_1, n_2, \ldots, n_i+1, n_j, \ldots, n_k$	$\bar{n}; i^+$
$n_1, n_2, \ldots, n_i-1, n_j, \ldots, n_k$	$\bar{n}; i^-$
$n_1, n_2, \ldots, n_i+1, n_j-1, \ldots, n_k$	$\bar{n}; i^+, j^-$

Using Stochastic Balance Equation :

Flow into state $\bar{n}$ = Flow out of state $\bar{n}$ and assuming that $C_i = 1$ (Single Server Node) and that $n_i \geq 1$ at each node, we obtain

$$\sum_{i=1}^{k} \gamma_i P_{\bar{n};i^-} + \sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \mu_i r_{ij} P_{\bar{n};i^+,j^-} + \sum_{i=1}^{k} \mu_i r_{i0} P_{\bar{n};i^+}$$

$$= \sum_{i=1}^{k} \mu_i (1 - r_{ii}) P_{\bar{n}} + \sum_{i=1}^{k} \gamma_i P_{\bar{n}} \longrightarrow \text{([illegible])}$$

$$\left[\text{where} \quad \sum_{i=1}^{k} \gamma_i P_{\bar{n};i^-} = \gamma_1 P_{\bar{n};1^-} + \gamma_2 P_{\bar{n};2^-} + \cdots + \gamma_k P_{\bar{n};k^-} \right.$$

$$\sum_{j=1}^{k}\sum_{\substack{i=1\\(i\neq j)}}^{k} \mu_i r_{ij}\, p_{\bar{n};i^+,j^-} = \Big(\mu_1 r_{12}\, p_{\bar{n};1^+,2^-} + \mu_1 r_{13}\, p_{\bar{n};1^+,3^-}$$

$$+ \ldots\ldots + \mu_1 r_{1k}\, p_{\bar{n};1^+,k^-}\Big) + \Big(\mu_2 r_{21}\, p_{\bar{n};2^+,1^-} + \mu_2 r_{23}\, p_{\bar{n};2^+,3^-}$$

$$+ \ldots\ldots + \mu_2 r_{2k}\, p_{\bar{n};2^+,k^-}\Big) + \ldots\ldots + \Big(\mu_k r_{k1}\, p_{\bar{n};k^+,1^-}$$

$$+ \mu_k r_{k2}\, p_{\bar{n};k^+,2^-} + \ldots\ldots + \mu_k r_{k(k-1)}\, p_{\bar{n};k^+,(k-1)^-}\Big)$$

$$\sum_{i=1}^{k} \mu_i r_{i0}\, p_{\bar{n};i^+} = \mu_1 r_{10}\, p_{\bar{n};1^+} + \mu_2 r_{20}\, p_{\bar{n};2^+} + \ldots\ldots + \mu_k r_{k0}\, p_{\bar{n};k^+}$$

$$\sum_{i=1}^{k} \mu_i (1-r_{ii})\, p_{\bar{n}} = \mu_1 (1-r_{11})\, p_{\bar{n}} + \mu_2 (1-r_{22})\, p_{\bar{n}} + \ldots + \mu_k (1-r_{kk})\, p_{\bar{n}}$$

$$\sum_{i=1}^{k} \gamma_k\, p_{\bar{n}} = \gamma_1\, p_{\bar{n}} + \gamma_2\, p_{\bar{n}} + \ldots\ldots + \gamma_k\, p_{\bar{n}}\Bigg]$$

Jackson showed that the solution to these steady state balance equations, is, amazingly of what has come to be generally called "product form".

Let λ_i be the total mean flow rate into node i (from outside and from other nodes)

Let γ_i = mean arrival to node i

r_{ij} = probability that a customer who has completed service at node j will go next to node i   i=1,2,3,...k ; j=0,1,2,...k

r_{i0} = probability that a customer will depart from the system from node i.

$$\lambda_i = \gamma_i + \sum_{j=1}^{k} r_{ji} \lambda_j \longrightarrow (1.26)$$

We define $\rho_i = \dfrac{\lambda_i}{\mu_i}$; $i = 1, 2, 3, \ldots, k$

Jackson showed that the steady state solution to Equation (1.25) is

$$P_{\bar n} = P(n_1, n_2, \ldots, n_k)$$

$$P_{\bar n} = (1-\rho_1)\rho_1^{n_1} (1-\rho_2)\rho_2^{n_2} \ldots (1-\rho_k)\rho_k^{n_k} \longrightarrow (1.27)$$

To shows (1.27) satisfies (1.25) we first show that

$$P_{\bar n} = C \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k}$$

Satisfied (1.25) where $C = \prod_{i=1}^{k} (1-\rho_i)$

We let $\quad R^{\bar n} = \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k}$

Then $P_{\bar n}$ becomes

$$P_{\bar n} = CR^{\bar n}$$

Now

$$P_{\bar n;i^-} = CR^{\bar n} \rho_i^{-1} = \frac{C}{\rho_i} R^{\bar n}$$

$$P_{\bar n;i^+j^-} = C\Re^{\bar n} \rho_i \rho_j^{-1} = \frac{C\rho_i}{\rho_j} \Re^{\bar n}$$

$$P_{\bar n;i^+} = C\Re^{\bar n} \rho_i$$

Substituting these values in Equation (1.25)

$$\mathfrak{R}^{\bar{n}} \sum_{i=1}^{k} \frac{\gamma_i}{\rho_i} = \mathfrak{R}^{\bar{n}} \sum_{\substack{j=1 \\ (i \neq j)}}^{k} \sum_{i=1}^{k} \mu_i \, r_{ij} \frac{\rho_i}{\rho_j}$$

$$+ \mathfrak{R}^{\bar{n}} \sum_{i=1}^{k} \mu_i \, r_{10} \, \rho_i = \mathfrak{R}^{\bar{n}} \sum_{i=1}^{k} \mu_i \, (1 - r_{ii})$$

$$- \mathfrak{R}^{\bar{n}} \sum_{i=1}^{k} \gamma_i$$

Cancelling out $\mathfrak{R}^{\bar{n}}$, we have

$$\sum_{i=1}^{k} \frac{\gamma_i \mu_i}{\lambda_i} + \sum_{\substack{j=1 \\ (i \neq j)}}^{k} \sum_{i=1}^{k} \mu_i \, r_{ij} \frac{\lambda_i \mu_j}{\mu_i \lambda_j} + \sum_{i=1}^{k} \mu_i r_{10} \frac{\lambda_i}{\mu_i}$$

$$= \sum_{i=1}^{k} \left[\mu_i \, (1 - r_{ii}) - \gamma_i \right] \longrightarrow (1.28)$$

From equation (1.27), we have

$$\lambda_j = \gamma_j + \sum_{\substack{i=1 \\ (i \neq j)}}^{k} r_{ij} \, \lambda_i + r_{jj} \, \lambda_j$$

$$\sum_{\substack{i=1 \\ (i \neq j)}}^{k} r_{ij} \, \lambda_i = \lambda_j - \gamma_j - r_{jj} \lambda_j \longrightarrow (1.29)$$

Substituting in Equation (1.28), we get

$$\sum_{i=1}^{k} \gamma_i \frac{\mu_i}{\lambda_i} + \sum_{j=1}^{k} \frac{\mu_j}{\lambda_j} (\lambda_j - \gamma_j - r_{jj} \lambda_j) + \sum_{i=1}^{k} \mu_i r_{10} \frac{\lambda_i}{\mu_i}$$

$$= \sum_{i=1}^{k} \left[\mu_i - \mu_i \, r_{ii} - \gamma_i \right]$$

Changing the subscript from j to i in IInd term of LHS

$$\sum_{i=1}^{k} \gamma_i \frac{\mu_i}{\lambda_i} + \sum_{i=1}^{k} \frac{\mu_i}{\lambda_i} (\lambda_i - \gamma_i - r_{ii}\lambda_i) + \sum_{i=1}^{k} \mu_i r_{10} \frac{\lambda_i}{\mu_i}$$

$$= \sum_{i=1}^{k} \left[\mu_i - \mu_i r_{ii} + \gamma_i \right]$$

$$\Rightarrow \sum_{i=1}^{k} \left[\gamma_i \frac{\mu_i}{\lambda_i} + \frac{\mu_i}{\lambda_i} (\lambda_i - \gamma_i - r_{ii}\lambda_i) + \mu_i r_{10} \frac{\lambda_i}{\mu_i} \right]$$

$$= \sum_{i=1}^{k} \left[\mu_i - \mu_i r_{ii} + \gamma_i \right]$$

$$\Rightarrow \sum_{i=1}^{k} \left[\mu_i (1-r_{ii}) + \lambda_i r_{10} \right] = \sum_{i=1}^{k} \left[\mu_i (1-r_{ii}) + \gamma_i \right]$$

$$\Rightarrow \sum_{i=1}^{k} \mu_i (1-r_{ii}) + \sum_{i=1}^{k} \lambda_i r_{10} = \sum_{i=1}^{k} \mu_i (1-r_{ii}) + \sum_{i=1}^{k} \gamma_i$$

$$\sum_{i=1}^{k} \lambda_i r_{10} = \sum_{i=1}^{k} \gamma_i$$

$$\boxed{\text{Total flow out of the network} = \text{Total flow in the Network}}$$

For steady-state, these must be equal. Now to evaluate C, we have

$$\sum_{n_k=0}^{\infty} \sum_{n_{k-1}=0}^{\infty} \text{-----} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} C\rho_1^{n_1} \rho_2^{n_2} \text{--} \rho_k^{n_k} = 1$$

$$C \left[\sum_{n_k=0}^{\infty} \rho_k^{n_k} \text{----} \sum_{n_2=0}^{\infty} \rho_2^{n_2} \sum_{n_1=0}^{\infty} \rho_1^{n_1} \right] = 1$$

$$C \left[\frac{1}{(1-\rho_k)} \frac{1}{(1-\rho_{k-1})} \text{------} \frac{1}{(1-\rho_2)} \frac{1}{(1-\rho_1)} \right] = 1$$

$$C = (1-\rho_1)(1-\rho_2) \cdots (1-\rho_k)$$

$$\boxed{C = \prod_{i=1}^{k} (1-\rho_i); \qquad \rho_i < 1, \; i=1,2,\ldots,k}$$

We can obtain expected measure rather easily for individual nodes since

$$L_i = \frac{\rho_i}{1-\rho_i} \quad \text{and} \quad W_i = \frac{L_i}{\lambda_i}$$

This is so because of the product form of the solution to the joint probability distribution and again does not imply that the nodes are truely M/M/1.

The above results for Jackson networks generalize easily to C-channel nodes.

Let C_i represents the number of servers at node i each having exponential service time with parameter μ_i. Then Equation (1.27) becomes

$$P_n = P(n_1, n_2, -- n_k) = \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)} P_{0i}$$

where

$$\rho_i = \frac{\lambda_i}{\mu_i}$$

and

$$a_i(n_i) = \begin{cases} (n_i)! & n_i \le C_i \\ C_i^{(n_i - C_i)} C_i! & n_i \ge C_i \end{cases}$$

and P_{0i} is such that

$$\sum_{n_i=1}^{\infty} P_{0i} \frac{\rho_i^{n_i}}{a_i(n_i)} = 1$$

1.5 CLOSED JACKSON NETWORKS :

In a closed Jackson network $\gamma_i = 0$ and $r_{i0} = 0$ for all i. Also we have a finite source queue of N(say) items which continuously travel inside the network.

By putting $\gamma_i = 0$ and $r_{i0} = 0$ in the equation (1.27) of open network model. We get

$$\sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \mu_i r_{ij} \, p_{\bar{n};\, i^+ j^-} = \sum_{i=1}^{k} \mu_i (1 - r_{ii}) p_{\bar{n}} \longrightarrow (1.30)$$

It also have a product from solution. The solution is again of the form

$$p_{\bar{n}} = C\rho_1^{n_1} \rho_2^{n_2} \text{------} \rho_k^{n_k} = C\Re^{\bar{n}} \longrightarrow (1.31)$$

Then

$$p_{\bar{n};\, i^+ j^-} = C\frac{\rho_i}{\rho_j} \Re^{\bar{n}}, \quad p_{\bar{n}} = C\Re^{\bar{n}}$$

Substitute these values in Eq. (1.30)

$$\sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \mu_i r_{ij} \, C\frac{\rho_i}{\rho_j} \Re^{\bar{n}} = \sum_{i=1}^{k} \mu_i (1 - r_{ii}) C\Re^{\bar{n}}$$

$$-\sum_{i=1}^{k} \mu_i r_{ii} + \sum_{j=1}^{k} \sum_{i=1}^{k} \mu_i r_{ij} \frac{\rho_i}{\rho_j} = \sum_{i=1}^{k} \mu_i - \sum_{i=1}^{k} \mu_i r_{ii}$$

$$\sum_{j=1}^{k} \frac{1}{\rho_j} \left[\sum_{i=1}^{k} \mu_i r_{ij} \rho_i \right] = \sum_{i=1}^{k} \mu_i \longrightarrow (1.32)$$

Using the flow into node i is equal to flow out of node i. We get

$$\mu_i \rho_i = \sum_{j=1}^{k} \mu_j r_{ji} \rho_j \longrightarrow (1.33)$$

Using Equation (1.33) in Equation (1.32),

$$\sum_{j=1}^{k} \frac{1}{\rho_j} \mu_j \rho_j = \sum_{i=1}^{k} \mu_i$$

$$\sum_{j=1}^{k} \mu_j = \sum_{i=1}^{k} \mu_i \longrightarrow (1.34)$$

that is an identity.

Now to evaluate C, we use

$$\sum_{n_1+n_2+..+n_k=N} C \rho_1^{n_1} \rho_2^{n_2} \text{------} \rho_k^{n_k} = 1$$

$$\Rightarrow \boxed{C = \left[\sum_{n_1+n_2+..+n_k=N} \rho_1^{n_1} \rho_2^{n_2} \text{------} \rho_k^{n_k} \right]^{-1}} \longrightarrow (1.35)$$

The constant C is shown as C(N), since it is a function of the total population size N. This can be written as

$$C^{-1}(N) = G(N)$$

so that

$$P_{n_1 n_2 .. n_k} = \frac{1}{G(N)} \rho_1^{n_1} \rho_2^{n_2} \text{------} \rho_k^{n_k} \longrightarrow (1.36)$$

where

$$G(N) = \sum_{n_1+n_2+..+n_k=N} \rho_1^{n_1} \rho_2^{n_2} \text{------} \rho_k^{n_k} \longrightarrow (1.37)$$

Again this closed network can easily be extended to arrivals at node 1. Then the solution becomes

$$P_{n_1, n_2 \ldots n_k} = \frac{1}{G(N)} \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)} \longrightarrow (1.38)$$

where

$$a_i(n_i) = \begin{cases} (n_i)! & n_i \leq c_i \\ c_i^{(n_i - c_i)} (c_i)! & n_i \geq c_i \end{cases}$$

and

$$G(N) = \sum_{n_1 + n_2 + \ldots + n_k = N} \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)}$$

.6 CYCLIC QUEUES :

If a closed network of k nodes such that

$$r_{ij} = \begin{cases} 1 & (j=i+1,\ 1 \leq i \leq (k-1)) \\ 1 & (i=k,\ j=1) \\ 0 & (\text{elsewhere}) \end{cases} \longrightarrow (1.39)$$

then we have a cyclic queue.

A cyclic queue is a sort of series queue in a circle where the output of the last node feeds back to the first node.

Hence for single servers at each node we have equations such that

$$P_{n_1, n_2 \ldots n_k} = C \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k} \longrightarrow (1.40)$$

where

$$\mu_i \rho_i = \sum_{j=1}^{k} \mu_j r_{ji} \rho_j$$

Using (1.37) in Equation (1.40), we get

$$\mu_i \rho_i = \begin{cases} \mu_{i-1}\rho_{i-1} & (i=2,3,\ldots,k) \\ \mu_k \rho_k & (i=1) \end{cases}$$

Thus we have

$$\rho_i = \begin{cases} \dfrac{\mu_{i-1}}{\mu_i}\,\rho_{i-1} & (i=2,3,\ldots,k) \\ \dfrac{\mu_k}{\mu_1}\,\rho_k & (i=1) \end{cases} \longrightarrow (1.41)$$

From Equation (1.41), we see that

$$\rho_2 = \frac{\mu_1}{\mu_2}\,\rho_1$$

$$\rho_3 = \frac{\mu_2}{\mu_3}\,\rho_2 = \frac{\mu_2}{\mu_3}\,\frac{\mu_1}{\mu_2}\,\rho_1 = \frac{\mu_1}{\mu_3}\,\rho_1$$

Similarly $\rho_4 = \dfrac{\mu_1}{\mu_4}\,\rho_1$

$$\rho_{k-1} = \frac{\mu_1}{\mu_{k-1}}\,\rho_1$$

$$\rho_k = \frac{\mu_1}{\mu_k}\,\rho_1$$

We select $\rho_1 = 1$ and substituting in (1.41), we obtain

$$P_{n_1,n_2,\ldots n_k} = \frac{1}{G(N)}\;\frac{\mu_1^{N-n_1}}{\mu_2^{n_2}\,\mu_3^{n_3}\cdots\mu_k^{n_k}} \longrightarrow (1.42)$$

where

$$G(N) = \sum_{n_1+n_2+\ldots+n_k=N} \rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k} \longrightarrow (1.43)$$

1.7 TRANSIENT SOLUTION OF QUEUEING SYSTEM M/M/1 -

In transient solution of queueing system, the system is time dependent. The equations for this system can be written as :

Let $P_n(t)$ = P {there are n units in the system at time t}

For $n \geq 1$

$$P_n(t+\Delta t) = P_n(t)\, P\{\text{no arrival and no service in } (t, t+\Delta t)\}$$
$$+ P_{n-1}(t)\, P\{\text{one arrival in } (t, t+\Delta t)\}$$
$$+ P_{n+1}(t)\, P\{\text{one service in } (t, t+\Delta t)\} + O(\Delta t)$$

$$P_n(t+\Delta t) = P_n(t)\, (1-\lambda\Delta t+O(\Delta t))\, (1-\mu\Delta t+O(\Delta t))$$
$$+ P_{n-1}(t)\, (\lambda\Delta t+O(\Delta t))\, (1-\mu\Delta t+O(\Delta t))$$
$$+ P_{n+1}(t)\, (\mu\Delta t+O(\Delta t))\, (1-\lambda\Delta t+O(\Delta t)) + O(\Delta t)$$

$$P_n(t+\Delta t) = P_n(t)\, [1-(\lambda+\mu)\Delta t] + P_{n-1}(t)\, \lambda\Delta t + P_{n+1}(t)\, \mu\Delta t$$
$$+ O(\Delta t)$$

$$P_n(t+\Delta t) - P_n(t) = -(\lambda+\mu)\Delta t\, P_n(t) + \lambda P_{n-1}(t)\Delta t + \mu P_{n+1}(t)\, \Delta t$$
$$+ O(\Delta t)$$

Dividing by Δt and taking $\Delta t \longrightarrow 0$, we have

$$\frac{d}{dt}[P_n(t)] = -(\lambda+\mu)\, P_n(t) + \lambda\, P_{n-1}(t) + \mu\, P_{n+1}(t)$$
$$, n \geq 1 \longrightarrow \quad (1.44)$$

For $n=0$

$$P_0(t+\Delta t) = P_0(t)\, P\,\{\text{no arrival in } (t, t+\Delta t)\}$$
$$+ P_1(t)\, P\,\{\text{one service in } (t, t+\Delta t)\} + o(\Delta t)$$

$$P_0(t+\Delta t) = P_0(t)\,(1-\lambda(\Delta t) + o(\Delta t))$$
$$+ P_1(t)\,(\mu\Delta t + o(\Delta t))\,(1-\lambda\Delta t + o(\Delta t))$$

$$P_0(t+\Delta t) - P_0(t) = -\lambda\Delta t P_0(t) + \mu\Delta t P_1(t) + o(\Delta t)$$

divided by Δt and taking $\Delta t \longrightarrow 0$

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t) \longrightarrow (1.45)$$

Let the probability generating function (p.g.f)

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t)\, z^n \longrightarrow (1.46)$$

Multiplying by z^n in Eq. (1.44), summing over n and adding (1.46), Then we get

$$\frac{d}{dt}\left[\sum_{n=1}^{\infty} P_n(t)\, z^n + P_0(t)\right] = -(\lambda+\mu)\sum_{n=1}^{\infty} P_n(t)\, z^n + \lambda \sum_{n=1}^{\infty} P_{n-1}(t)\, z^n$$
$$+ \mu\sum_{n=1}^{\infty} P_{n+1}(t) z^n - \lambda P_0(t) + \mu P_1(t)$$

$$\frac{d}{dt}\left[\sum_{n=0}^{\infty} P_n(t) z^n\right] = -(\lambda+\mu)\sum_{n=0}^{\infty} P_n(t)\, z^n + (\lambda+\mu) P_0(t)$$
$$+ \lambda z \sum_{n=1}^{\infty} P_{n-1}(t) z^{n-1} + \frac{\mu}{z}\sum_{n=1}^{\infty} P_{n+1}(t) z^{n+1}$$
$$-\lambda P_0(t) + \mu P_1(t)$$

$$\frac{d}{dt}\left[G(z,t) \right] = -(\lambda+\mu)\ G(z,t) + \lambda z \sum_{n=0}^{\infty} P_n(t) z^n + \frac{\mu}{z} \sum_{n=2}^{\infty} P_n(t) z^n$$
$$+ \mu P_0(t) + \mu P_1(t)$$

$$\frac{d}{dt}\left[G(z,t) \right] = -(\lambda+\mu)\ G(z,t) + \lambda z\ G(z,t) + \frac{\mu}{z} \sum_{n=0}^{\infty} P_n(t) z^n$$
$$+ \frac{\mu}{z}\left[-P_0(t) - P_1(t) z \right] + \mu P_0(t) + \mu P_1(t)$$

$$\frac{d}{dt}\left[G(z,t) \right] = -(\lambda+\mu-\lambda z)\ G(z,t) + \frac{\mu}{z}\ G(z,t) + \mu\left(1 - \frac{1}{z} \right) P_0(t)$$

$$\frac{d}{dt}\left[G(z,t) \right] = -\left(\lambda + \mu - \lambda z - \frac{\mu}{z}\right)\ G(z,t) + \mu\left(1 - \frac{1}{z}\right) P_0(t) \rightarrow (1.47)$$

Let the system start at time t=0 with i units in the system.

Put t=0 in Equation (1.46), we get

$$G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

$$\text{But } P_n(0) = \delta_{in} = \begin{cases} 1 & \text{when } n = i \\ 0 & \text{when } n \neq i \end{cases} \longrightarrow (1.48)$$

Then n = i

$$G(z,0) = P_i(0) z^i$$
$$G(z,0) = \delta_{ii} z^i$$
$$G(z,0) = z^i$$

Where δ_{in} is called the Kronecker delta.

Let $g(z,s)$ be the Laplace Transform (L.T.) of $G(z,t)$

$$L\left[G(z,t)\right] = g(z,s) = \int_0^\infty e^{-st}\, G(z,t)\, dt \longrightarrow (1.49)$$

Taking Laplace Transform on both sides of equation (1.47),

$$L\left(\frac{d}{dt} G(z,t)\right) = -L\left[\left(\lambda + \mu - \lambda z - \frac{\mu}{z}\right) G(z,t)\right] + L\left[\mu\left(1 - \frac{1}{z}\right) P_0(t)\right]$$

$$\int_0^\infty e^{-st}\left[\frac{d}{dt} G(z,t)\right] dt = -\left(\lambda+\mu-\lambda z-\frac{\mu}{z}\right) L\left[G(z,t)\right] + \mu\left(1-\frac{1}{z}\right) L\left[P_0(t)\right]$$

$$\left[e^{-st} G(z,t)\right]_0^\infty - \int_0^\infty (-s) e^{-st}\, G(z,t)\, dt$$

$$= -(\lambda+\mu-\lambda z-\frac{\mu}{z}) g(z,s) + \mu(1-\frac{1}{z}) p_0(s)$$

$$-G(z,0) + s\int_0^\infty e^{-st}\, G(z,t)\,dt = -(\lambda+\mu-\lambda z-\frac{\mu}{z}) g(z,s) + \mu(1-\frac{1}{z}) p_0(s)$$

$$-z^i + s g(z,s) = -(\lambda+\mu-\lambda z-\frac{\mu}{z}) g(z,s) + \mu(1-\frac{1}{z}) p_0(s)$$

$$(s+\lambda+\mu-\lambda z-\frac{\mu}{z}) g(z,s) = z^i + \mu(1-\frac{1}{z}) p_0(s) \longrightarrow (1.50)$$

$$g(z,s) = \frac{[z^{i+1}+\mu(z-1)p_0(s)]}{[(s+\lambda+\mu)z-\lambda z^2-\mu]} \longrightarrow (1.51)$$

To find the zeros of denominator, we put

$$(s+\lambda+\mu)z-\lambda z^2-\mu = 0$$

$$\lambda z^2-(s+\lambda+\mu)z+\mu = 0$$

$$z = \frac{(s+\lambda+\mu) \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}$$

Let it has two roots $\alpha_1(s)$ and $\alpha_2(s)$. Hence

$$\alpha_1(s) = (2\lambda)^{-1}\left[(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\right] \longrightarrow (1.52)$$

$$\alpha_2(s) = (2\lambda)^{-1}\left[(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\right] \longrightarrow (1.53)$$

Now, let

$$f(z) = (s+\lambda+\mu)z \quad , \quad g(z) = -(\lambda z^2+\mu) \longrightarrow (1.54)$$

Then on the unit circle $|z|=1$

$$|f(z)| = |s+\lambda+\mu| > |\lambda+\mu| \geq |g(z)|$$

Hence $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the unit circle. But $f(z)$ has only one zero, and so $f(z)+g(z)$ also has only one zero inside the unit circle. This zero will be α_1. Thus we have

$$z^{i+1} - \mu(1-z)p_0(s) = 0$$

$$\mu(1-z)p_0(s) = z^{i+1}$$

$$p_0(s) = \frac{z^{i+1}}{\mu(1-z)}$$

But root of z is α_1. Therefore

$$p_0(s) = \frac{\alpha_1^{i+1}}{\mu(1-\alpha_1)} \longrightarrow (1.55)$$

By Equation (1.51)

$$g(z,s) = \frac{[z^{i+1}-\mu(1-z)p_0(s)]}{-\lambda(z^2-(s+\lambda+\mu)/\lambda\ z+\mu/\lambda)}$$

$$g(z,s) = \frac{z^{i+1}-\lambda(1-z)\alpha_1^{i+1}/\lambda(1-\alpha_1)}{-\lambda(z-\alpha_1)(z-\alpha_2)} \longrightarrow (1.56)$$

$$g(z,s) = -\frac{z^{i+1}(1-\alpha_1) - (1-z)\alpha_1^{i+1}}{\lambda(z-\alpha_1)(z-\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{(z^{i+1}-\alpha_1^{i+1}) - z\alpha_1(z^i-\alpha_1^i)}{\lambda\alpha_2(z-\alpha_1)(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{(z-\alpha_1)(z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i) - z\alpha_1(z-\alpha_1)(z^{i-1}+\alpha_1 z^{i-2}+\ldots+\alpha_1^{i-1})}{\lambda\alpha_1(z-\alpha_1)(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{(z-\alpha_1)\left[z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i - z^i\alpha_1 - z^{i-1}\alpha_1^2 - \ldots - z\alpha_1^i\right]}{\lambda\alpha_2(z-\alpha_1)(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{z^i(1-\alpha_1)+\alpha_1 z^{i-1}(1-\alpha_1)+\ldots+\alpha_1^i(1-\alpha_1)+\alpha_1^{i+1}}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{(1-\alpha_1)(z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i)}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{(1-\alpha_1)(z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i)}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)} + \frac{\alpha_1^{i+1}}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)}$$

$$g(z,s) = \frac{1}{\lambda\alpha_2}(z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i)(1-z/\alpha_2)^{-1} + \frac{\alpha_1^{i+1}}{\lambda\alpha_2(1-\alpha_1)}(1-z/\alpha_2)^{-1}$$

By Binominal expansion, we can write

$$\left(1-\frac{z}{\alpha_2}\right)^{-1} = \sum_{k=0}^{\infty}\left(\frac{z}{\alpha_2}\right)^k$$

$$\therefore g(i,z) = \frac{1}{\lambda\alpha_2}\left(z^i+\alpha_1 z^{i-1}+\ldots+\alpha_1^i\right)\sum_{k=0}^{\infty}\left(\frac{z}{\alpha_2}\right)^k$$

$$+\frac{\alpha_1^{i+1}}{\lambda\alpha_2(1-\alpha_1)}\sum_{k=0}^{\infty}\left(\frac{z}{\alpha_2}\right)^k \longrightarrow (1.57)$$

Now $p_n(s)$, the Laplace Transform of $P_n(t)$ is the coefficient of z^n in $g(i,z)$.

The coefficient of z^n from the second term on the right hand side of Equation (1.57)

$$= \frac{\alpha_1^{i+1}}{\lambda\alpha_2(1-\alpha_1)} \quad \frac{1}{\alpha_2^n}.$$

$$= \frac{\alpha_1^{i+1}}{\lambda\alpha_2^{n+1}(1-\alpha_1)} = \frac{\alpha_1^{i+1}}{\lambda\alpha_2^{n+1}}(1-\alpha_1)^{-1}$$

$$= \frac{\alpha_1^{i+1}}{\lambda\alpha_2^{n+1}}\left(1+\alpha_1+\alpha_1^2+\cdots\cdots\right)$$

$$= \frac{1}{\lambda}\quad\frac{\alpha_1^{i+1}\ \alpha_1^{n+1}}{\alpha_2^{n+1}\ \alpha_1^{n+1}}\sum_{k=0}^{\infty}(\alpha_1)^k$$

$$= \frac{1}{\lambda}\quad\frac{\alpha_1^{n+i+2}}{(\alpha_1\alpha_2)^{n+1}}\sum_{k=0}^{\infty}(\alpha_1)^k$$

$$= \frac{1}{\lambda}\quad\frac{1}{(\alpha_1\alpha_2)^{n+1}}\sum_{k=n+i+2}^{\infty}\alpha_1^k$$

But $\qquad \alpha_1\alpha_2 = \frac{\mu}{\lambda} \qquad \Rightarrow \alpha_1 = \left(\frac{\mu}{\lambda\alpha_2}\right)$

Therefore of Coefficient of z^n from the second term in RHS

$$\frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\left(\frac{\mu}{\lambda}\right)^{k}\frac{1}{\alpha_2^{k}} \longrightarrow (1.58)$$

The first term on RHS can be written as

$$\frac{1}{\lambda\alpha_2}\left(z^{i}+\alpha_1 z^{i-1}+\dots+\alpha_1^{i}\right)\sum_{k=0}^{\infty}\left(\frac{z}{\alpha_2}\right)^{k}$$

$$=\frac{1}{\lambda\alpha_2}\left[z^{i}+\alpha_1 z^{i-1}+\dots+\alpha_1^{i-n}z^{n}+\alpha_1^{i-n+1}z^{n-1}+\alpha_1^{i-n+2}z^{n-2}+\dots+\alpha_1^{i}\right]$$

$$\times\left[1+\frac{z}{\alpha_2}+\frac{z^2}{\alpha_2^{2}}+\dots+\frac{z^n}{\alpha_2^{n}}+\dots\right]$$

The coefficient of z^n

$$=\frac{1}{\lambda\alpha_2}\left[\frac{\alpha_1^{i-n}}{1}+\frac{\alpha_1^{i-n+1}}{\alpha_2}+\frac{\alpha_1^{i-n+2}}{\alpha_2^{2}}+\dots+\frac{\alpha_1^{i}}{\alpha_2^{n}}\right]$$

$$=\frac{1}{\lambda\alpha_2}\sum_{m=(i-n)^+}^{i}\frac{\alpha_1^{m}}{\alpha_2^{(n-i+m)}}=\sum_{m=(i-n)^+}^{i}\frac{1}{\lambda\alpha_2}\frac{(\mu/\lambda\alpha_2)^{m}}{\alpha_2^{(n-i+m)}}$$

The coefficient of z^n

$$=\sum_{m=(i-n)^+}^{i}\frac{(\mu/\lambda)^{m}}{\lambda\alpha_2^{(n-i+2m+1)}} \longrightarrow (1.59)$$

The coefficient of z^n of Equation (1.59) can be written as

$$r_n(i)=\frac{1}{\lambda}\left[\sum_{m=(i-n)^+}^{i}\frac{(\mu/\lambda)^{m}}{\alpha_2^{(n-i+2m+1)}}+\left(\frac{\lambda}{\mu}\right)^{i+1}\sum_{k=n+i}^{\infty}\frac{(\mu/\lambda)^{k}}{\alpha_2^{k}}\right] \rightarrow (1.60)$$

For the Inverse Laplace Transform

$$L^{-1}\left[\left(s-\sqrt{s^2-a^2}\right)^{\nu}\right]=\nu a^{\nu}t^{-1}I_{\nu}(at) \longrightarrow (1.61)$$

Where $I_\nu(z)$ is the modified Bessel function of the first kind and order ν, given

$$I_{\nu}(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \longrightarrow (1.62)$$

Therefore

$$L^{-1}\left\{\frac{1}{\alpha_2^k}\right\} = L^{-1}\left(\alpha_2^{-k}\right)$$

$$= L^{-1}\left[(2\lambda)^{k}\left\{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\right\}^{-k}\right]$$

$$= (2\lambda)^{k}\, L^{-1}\left[\left\{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\right\}^{-k}\right]$$

$$= (2\lambda)^{k}\, e^{-(\lambda+\mu)t}\, L^{-1}\left[\left\{s - \sqrt{s^2-(2\sqrt{\lambda\mu})^2}\right\}^{-k}\right]$$

$$= (2\lambda)^{k}\, e^{-(\lambda+\mu)t}\, k(2\sqrt{\lambda\mu})^{-k}\, t^{-1}\, I_k(2\sqrt{\lambda\mu}\,t)$$

$$= 2^k\lambda^{k}\, e^{-(\lambda+\mu)t}\, 2^{-k}\, \lambda^{-k/2}\, \mu^{-k/2}\, t^{-1}\, k\, I_k(2\sqrt{\lambda\mu}\,t)$$

$$= e^{-(\lambda+\mu)t}\left[\sqrt{\lambda/\mu}\right]^{k} t^{-1}\, k\, I_k(2\sqrt{\lambda\mu}\,t)$$

$$L^{-1}\left\{\frac{1}{\alpha_2^k}\right\} = e^{-(\lambda+\mu)t}\left[\sqrt{\lambda/\mu}\right]^{k}\frac{k}{t}\, I_k(2\sqrt{\lambda\mu}\,t) \longrightarrow (1.63)$$

Similarly we can write

$$L^{-1}\left[\frac{1}{\alpha_{2}^{(n-i+2m+1)}}\right] = e^{-(\lambda+\mu)t}\left(\sqrt{\lambda/\mu}\right)^{n-i+2m+1}$$

$$\times\ \frac{(n-i+2m+1)}{t}\ I_{(n-i+2m+1)}(2\sqrt{\lambda\mu}\ t) \longrightarrow (1.64)$$

Taking the Inverse Laplace Transform of Equation

$$L^{-1}\left\{P_n(s)\right\} = L^{-1}\left[\frac{1}{\lambda}\left[\sum_{m=(i-n)^+}^{i}\frac{(\mu/\lambda)^m}{\alpha_{2}^{n-i+2m+1}} + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\frac{(\mu/\lambda)^k}{\alpha_{2}^{k}}\right]\right]$$

$$P_n(t) = \frac{1}{\lambda}\sum_{m=(i-n)^+}^{i}\left(\frac{\mu}{\lambda}\right)^m L^{-1}\left[\frac{1}{\alpha_{2}^{n-i+2m+1}}\right] + \frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\left(\frac{\mu}{\lambda}\right)^k L^{-1}\left[\frac{1}{\alpha_{2}^{k}}\right]$$

Substituting the values of Inverse Laplace Transform

$$P_n(t) = \frac{1}{\lambda}\sum_{m=(i-n)^+}^{i}\left(\frac{\mu}{\lambda}\right)^m e^{-(\lambda+\mu)t}\left(\sqrt{\lambda/\mu}\right)^{n-i+2m+1}$$

$$\times\left(\frac{n-i+2m+1}{t}\right) I_{(n-i+2m+1)}(2\sqrt{\lambda\mu}\,t)$$

$$+\frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\left(\frac{\mu}{\lambda}\right)^k e^{-(\lambda+\mu)t}\left(\sqrt{\lambda/\mu}\right)^k \frac{k}{t}\ I_k\ (2\sqrt{\lambda\mu}\ t)$$

$$P_n(t) = \frac{e^{-(\lambda+\mu)t}}{\lambda} \left[\left(\sqrt{\lambda/\mu}\right)^{n-i+1} \sum_{m=(i-n)^+}^{i} \left(\frac{n-i+2m+1}{t}\right) \lambda\, I_{(n-i+2m+1)}(2\sqrt{\lambda\mu}\,t) \right.$$

$$\left. \left(\frac{\lambda}{\mu}\right)^{n+1} \sum_{k=n+i+2}^{\infty} \left(\sqrt{\mu/\lambda}\right)^{k} \frac{k}{t}\, I_k\,(2\sqrt{\lambda\mu}\;t) \right] \longrightarrow (1.65)$$

On using recurrence relation

$$\left(\frac{2\nu}{z}\right) I_\nu(z) = I_{\nu-1}(z) - I_{\nu+1}(z) \longrightarrow (1.66)$$

In equations (1.65), we get

$$P_n(t) = e^{-(\lambda+\mu)t} \left[\left(\sqrt{\mu/\lambda}\right)^{i-n} I_{|n-i|} + \left(\sqrt{\mu/\lambda}\right)^{i-n+1} I_{n+i+1} \right.$$

$$\left. + \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^{n} \sum_{k=n+i+2}^{\infty} \left(\sqrt{\mu/\lambda}\right)^{k} I_k \right]$$

where $I_\nu = I_\nu\,(2\sqrt{\lambda\mu}\;t)$

1.8 DISCERTE TIME TRANSIENT SOLUTION :

Several queueing problems have been solved using steady state conditions. As compared to these problems, it seems that not much have been done to obtain the corresponding transient solutions. This is because of the fact that the transient solutions are not only mathematically intractable or excessively labourious but also computationally very costly. Therefore, we can say that most

of queueing theory results has concentrated on steady state solution or some approximations. In most of the cases even steady state solutions are difficult to compute. Chaudhry, M.L., Kapur, P.K., and Templeton, J.G.C. (1991, 1992) have set a new trend in the numerical computations of models in queue through the technique of using roots. Closed form solutions as well as exact computational results are obtained by this approach.

Tackac's (1962) gives two solutions for the $M/M/1/\infty$ neither of which easy to compute. The first solution is in terms of integrals whereas the second involves an infinite sums of Bessel functions.

The solution becomes a bit simpler if the waiting space is finite which may be true in many applications. In their case arrival and service rates are constant. Besides this they make use of spectral decomposition which require to find left and right eigen vectors. This is not easy if the matrix is very large.

The methods developed by Chaudhry, M.L., Kapur, P.K., and Templeton, J.G.C. (1991) avoides spectral decomposition and well suited method for small and large matrices. Numerical techniques such as Ranga Kutta, Euleur, Taylor and Randomization have been used to find transient solution.

Whereas first three have been employed in solving differential equation, the late one is particular suited for solving queueing problems. In order to get greater accuracy one needs to increase the number of steps. This together the number of simultaneous equations to be solved slows down the solution process considerably. Recently Sharma and Das (1988) have obtained transient solution to a special category of Markovian model using eigen values of matrices in queueing theory. Standard numerical packages are used to obtained eigen values, which are difficult to obtained when the matrices are large and a result computational difficulties arises.

Chaudhry, M.L., Kapur,P.K., and Templeton, J.G.C.(1991) have made attempt to obtained similar results in discrete time for finite waiting space problems in queueing theory. Since the transient solution depend on the initial state of system, it is intersecting to know the effect on the system behavior. Kobayashi (1982) has discussed the several system which operate at discrete time many machine cycling of a processor and several other examples in Computer Science.

Chaudhry, M.L., Kapur,P.K., and Templeton, J.G.C.(1991) give the transient solutions for a general class discrete time models in queueing theory. In this they

have assumed that the queue consists of finite waiting space, Interarrival and service probabilities are dependent on the state of the system, the interarrival and service time distributions are geometric but independent of time and queue discipline is first in first out.

By using these assumptions finally they make the difference equations. To solve these equations they use the method of probability generating function. Write these equations in the Matrix form. Cramer Rule has been applied for finding the solutions of these equations. Explicit closed form expressions for distributions has been obtained in terms of the roots of a characteristic equation.

To find the eigen values or characteristics roots they make use of QROOT software package which is developed at Royal Military College at Canada by M.L. Chaudhry (1992).

For the analysis of the model the following notations are used.

X_k - Number of Customers in queue at epoch k.

N - Size of waiting space.

λ_n - Interarrival probability when n customers are in system.

μ_n - Service probability when n customers are in the system.

$$\phi_n = \mu_n(1-\lambda_n)$$

$$\varphi_n = \lambda_n(1-\mu_n)$$

Here $P_m(n)$ denote the probability that the system is in the n^{th} state at the beginning of m^{th} epoch. X_m, $m\geq 0$ is an integer valued discrete stochastic process taking value $(0,1,2,\dots,N)$. $X_m = n$ $(0\leq n\leq N)$ implies that there are n customers in the system at epoch m. The difference equations are

$$P_{m+1}(0) - P_m(0) = -\psi_0 P_m(0) + \phi_1 P_m(1) \longrightarrow \quad (1.67)$$

$$P_{m+1}(n) - P_m(n) = P_m(n)\,(-\phi_n-\psi_n) + P_m(n-1)\psi_{n-1} + P_m(n+1)\phi_{n+1}$$

$$, \; 1 \leq n \leq (N-1) \longrightarrow \quad (1.68)$$

$$P_{m+1}(N) - P_m(N) = -\phi_N P_m(N) + P_m(N-1)\,\psi_{N-1} \longrightarrow \quad (1.69)$$

and $\quad P_0(i) = 1 \qquad 0 \leq i \leq N$

Let $P_z(n)$ be the p.g.f. of $P_m(n)$ defined as

$$P_z(n) = \sum_{m=0}^{\infty} Z^m P_m(n) \qquad |Z| \leq 1$$

Now taking the p.g.f. of Equation (1.67),(1.68) and (1.69),

we have

$$Ap = \begin{bmatrix} \delta_{k0} & \delta_{k1} & \dots & \delta_{kN} \end{bmatrix}' \longrightarrow \quad (1.70)$$

where A is a real tridiagonal (N+1)x(N+1) matrix, p is column vector and δ_{ki} is the Kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z & k=i \\ 0 & \text{otherwise} \end{cases}$$

Defining $(1-Z)/Z = s$ and assuming $\mu_0=0$ and $\lambda_N=0$

$$\begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 \quad \cdots & (N-2) & (N-1) & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ N-1 \\ N \end{array} & A(s) = \left[\begin{array}{cccccc} s+\psi_0 & -\phi_1 & 0 \; \cdots & 0 & 0 & 0 \\ -\psi_0 & s+\phi_1+\psi_1 & -\phi_2 \; \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot \; \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \; \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \; \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \; \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \; \cdots & -\psi_{N-2} & s+\phi_{N-1}+\psi_{N-1} & -\phi_N \\ 0 & 0 & 0 \; \cdots & 0 & -\psi_{N-1} & s+\phi_N \end{array}\right]_{(N+1)\times(N+1)} \end{array} \longrightarrow (1.71)$$

and

$$F = \begin{bmatrix} F_z(0) \\ \vdots \\ F_z(N) \end{bmatrix}_{(N+1)\times 1} \longrightarrow (1.72)$$

From (1.70) using Cramer's Rule $F_z(n)$ are explicitly determined as:

$$F_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|} \quad ; \quad |Z|\leq 1 \longrightarrow (1.73)$$

Applying some row and column transformations on $|A(s)|$ it may be expressed as $s|D(s)|$, where $D(s)$ is the matrix of order $N\times N$ given by

$$
D(s) = \begin{bmatrix}
s+(\phi_1+\lambda) & -\sqrt{\psi}\,\phi_1 & 0 & 0 & 0 & 0 \\
-\sqrt{\psi}\,\phi_1 & s+\psi+(\phi_1+\phi_2) & -\sqrt{\psi(\phi_1+\phi_2)} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\sqrt{\psi(\phi_1+\phi_2)} & s+\psi+(\phi_1+\phi_2) & -\sqrt{\psi}\,\phi_1 \\
0 & 0 & 0 & 0 & -\sqrt{\psi}\,\phi_1 & s+\psi+(\mu_1+\mu_2)
\end{bmatrix}
$$

If D(s) expanded it will be a polynomial of degree N. The roots of the polynominal $|A(s)|$ are real, negative and distinct (one root being zero). Let α_k $(k=0,1,\dots N)$ be the root of $|A(s)|$ with $\alpha_0=0$, then

$$|A(s)| = s\prod_{k=1}^{N}(s-\alpha_k)$$

and hence $$F_i(n) = \frac{|A_{n+1,i}(s)|}{s\prod_{j=1}^{N}(s-\alpha_j)} \qquad 0 \le n \le N \longrightarrow (1.74)$$

Resolving the Right Hand Side of $F_i(n)$ into partial fractions replacing s by $(1-z)/z$, using initial conditions and comparing coefficient of z^m, we have

$$F_i(n) = b_n + \begin{cases} \prod_{r=n}^{(i-1)} \phi_{r+1} \sum_{k=1}^{N} \alpha_{kr}(1+\alpha_k)^m & 0\le n<i \\ \sum_{k=1}^{N} \alpha_{kn} (1+\alpha_k)^m & n=i \\ \prod_{r=i}^{n-1} \psi_r \sum_{k=1}^{N} \alpha_{kn} (1+\alpha_k)^m & i<n\le N \end{cases} \longrightarrow (1.75)$$

Where $\alpha_{k,n}$ and $b_{n,i}$ are defined as

$$\alpha_{nk} = \begin{cases} \dfrac{C_{N-i}(\alpha_k)D_n(\alpha_k)}{\alpha_k \prod_{j=1}^{N} (\alpha_k-\alpha_j)} & 0 \le n < 1 \\[2ex] \dfrac{C_{N-n}(\alpha_k)D_n(\alpha_k)}{\alpha_k \prod_{j=1}^{N} (\alpha_k-\alpha_j)} & n=1 \\[2ex] \dfrac{C_{N-n}(\alpha_k)D_n(\alpha_k)}{\alpha_k \prod_{\substack{j=1 \\ j\neq k}}^{N} (\alpha_k-\alpha_j)} & 1 < n \le N \end{cases}$$

$$\text{and } b_n = \frac{C_{N-n}(0)D_n(0)}{\prod_{k=1}^{N} (-\alpha_k)} \qquad 0 \le n \le N \longrightarrow (1.76)$$

Where $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left (nxn) square matrices formed from A(s) such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

For convenience, we write $C_n(s)$ and $D_n(s)$ as C_n and D_n respectively. Then

$C_1 = s+\phi_N$, $C_0 = 1$, $D_1 = s+\psi_0$, $D_0 = 1$ and $\mu_0 = \lambda_N = 0$

The C_i's and D_i's are given by

$$C_i = \left[s+\phi_{N+1-i}+\psi_{N+1-i}\right] C_{i-1} - \left[\phi_{N+2-i}\,\psi_{N+1-i}\right] C_{i-2}$$

$$2 \le i \le (N+1) \longrightarrow (1.77)$$

and

$$D_i = \left[s - \phi_{i-1} + \psi_{i-1} \right] D_{i-1} - \left[\phi_{i-1} \psi_{i-2} \right] D_{i-2}$$

$$2 \le i \le (N+1) \longrightarrow (1.78)$$

Using ORCOT Software package, we find the root α_k called the characteristic equation of A(s). After finding the roots they discuss many cases and find the numerical results.

CHAPTER TWO

DISCRETE TIME TRANSIENT SOLUTION FOR Geom(n)/Geom(n)/2/N WITH HETEROGENEOUS SERVER

2.1 INTRODUCTION :

In this chapter attempt has been made to obtain a discrete time transient solution of the model Geom(n)/Geom(n)/2/N with heterogeneous server. We assume that the inter arrival probabilities and service time probabilities of first and second servers to be geometrically distributed with parameters λ, μ_1 and μ_2 respectively. We also assume that $\mu_1 < \mu_2$ that is the service time probability for first server is less than that of second server. Which further implies that we are considering modified queue discipline i.e. the first arriving unit from amongst the initial number of unit present at the start of the service joins the first counter for service.

Therefore the arriving unit goes to the counter which it find free. The maximum number of customers in the system is restricted to N. We further assume that there is no unit initial waiting at the time t=0 when the service starts.

2.2. ASSUMPTIONS :

1. The queue consists of finite waiting space.
2. Inter arrival probabilities and service probabilities does not depend on the state of the system.
3. The interarrival and service time distribution are geometric but independent of time.
4. λ is the interarrival probability of a customer in the system and μ_1 and μ_2 be the service probability of a customer for server 1 and server 2 respectively such that $\mu_1 < \mu_2$ i.e. probability that a customer is serviced at server one is less than that of server two.
5. Queue discipline is First In First Out (FIFO).

2.3 NOTATION :

X_t : denote the number of customer at epoch t.

N : Size of waiting space.

λ : Interarrival probability of a customer.

μ_1 : Service probability of a customer for server one.

μ_2 : Service probability of a customer for server two.

ϕ_1 : $\mu_1(1-\lambda)$

ϕ_2 : $\mu_2(1-\lambda)$

ψ : $\lambda(1-\mu_1-\mu_2)$

2.4 ANALYSIS OF THE MODEL :

Let $P_m(n)$ $(n=0,1,2,\ldots N)$ denote the probability that the system is in the n^{th} state at the beginning of the m^{th} epoch or time slot. Let X_k be the number of customers in the system at discrete time epoch k. Then X_k, $k\geq 0$ is an integer valued discrete stochastic process taking values $0,1,2,\ldots N$. $X_k = n$ $(0\leq n\leq N)$ implies that there are n customers in the system at epoch k.

The following difference can be written as

$$P_{m+1}(0)-P_m(0) = -\lambda P_m(0) + \phi_1 P_m(1) \longrightarrow (2.1)$$

$$P_{m+1}(1)-P_m(1) = -P_m(1)(\psi+\phi_1+\phi_2)+\lambda P_m(0)+(\phi_1+\phi_2)P_m(2) \rightarrow (2.2)$$

$$P_{m+1}(n)-P_m(n) = -(\psi+\phi_1+\phi_2)P_m(n)+\psi P_m(n-1)+(\phi_1+\phi_2)P_m(n+1)$$
$$2\leq n\leq (N-2) \longrightarrow (2.3)$$

$$P_{m+1}(N-1)-P_m(N-1) = -(\psi+\phi_1)P_m(N-1)+\psi P_m(N-2)$$
$$+(\mu_1+\mu_2)P_m(N) \longrightarrow (2.4)$$

$$P_{m+1}(N)-P_m(N) = -(\mu_1+\mu_2)P_m(N) + \psi P_m(N-1) \longrightarrow (2.5)$$

where $P_0(i)=1$ $\quad 0\leq i\leq N$

Let $P(n)$ be the steady state distribution

i.e., $\quad \lim_{m\to\infty} P_m(n) = P(n)$

Let $F_z(n)$ be the probability generating function (p.g.f.) of $F_n(n)$ defined as

$$\Pi(z,t) = F_z(n) = \sum_{n=0}^{\infty} F_n(n) z^n \qquad |z| \le 1$$

Taking the p.g.f. of equation (2.1) to (2.5). For this we multiply the equation by z^n and taking summation from 0 to ∞ for n and using $\frac{(1-z)}{z} = s$, we get

$$(s-\lambda)F_z(0) - \phi_1 F_z(1) = 1/z \longrightarrow (2.6)$$

$$-\lambda F_z(0) + (s+\psi+\phi_1+\phi_2)F_z(1) - (\phi_1+\phi_2)F_z(2) = 1/z \longrightarrow (2.7)$$

$$-\psi F_z(n-1) + (s+\psi+\phi_1+\phi_2)F_z(n) - (\phi_1+\phi_2)F_z(n+1) = 1/z$$

$$;\ 2 \le n \le (N-2) \longrightarrow (2.8)$$

$$-\psi F_z(N-2) + (s+\psi+\phi_1)F_z(N-1) - (\mu_1+\mu_2)F_z(N) = 1/z \longrightarrow (2.9)$$

$$-\psi F_z(N-1) + (s+\mu_1+\mu_2)F_z(N) = 1/z \longrightarrow (2.10)$$

These equations (2.6) to (2.10) can be written as in the matrix form

$$AF = \left[\delta_{10}\ \delta_{11}\ \ldots\ldots\ \delta_{1N} \right]' \longrightarrow (2.11)$$

Where A is a real tridiagonal (N+1)x(N+1) matrix, F is a column vector of order (N+1)x1 and δ_{ki} is the Kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z & k=i \\ 0 & \text{otherwise} \end{cases}$$

[illegible]

$$
A(s) = \begin{array}{c c}
 & \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & (N-2) & (N-1) & N \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ (N-2) \\ (N-1) \\ N \end{array} &
\begin{bmatrix}
(s+\lambda) & -\phi_1 & 0 & \cdots & 0 & 0 & 0 \\
-\lambda & (s+\psi+\phi_1+\phi_2) & -(\phi_1+\phi_2) & \cdots & 0 & 0 & 0 \\
0 & -\psi & (s+\psi+\phi_1+\phi_2) & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & (s+\psi+\phi_1+\phi_2) & -(\phi_1+\phi_2) & 0 \\
0 & 0 & 0 & \cdots & -\psi & (s+\psi+\phi_1+\phi_2) & -(\mu_1+\mu_2) \\
0 & 0 & 0 & \cdots & 0 & -\psi & (s+\mu_1+\mu_2)
\end{bmatrix}
\end{array}
$$

and $F = \left[F_z(0) \quad F_z(1) \; \ldots\ldots\ldots \; F_z(N) \right]'$

From equation (2.11), using Cramer's Rule $F_z(n)$ are explicitly determined as

$$F_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|} \qquad 0 \le n \le N$$

Where $A_{n+1}(s)$ is obtained from $A(s)$ by replacing the $(n+1)^{th}$ column of $A(s)$ by the right hand side of equation (2.11) and $|A(s)|$ is the determinant of $A(s)$.

Applying some row and column transformations on $|A(s)|$, it may be expressed as $-|D(s)|$ is a real symmetric, tridiagonal matrix of order (NxN).

$$D(s)=\begin{array}{cc}
 & \begin{array}{ccccccc} 1 & 2 & 3 & & (N-2) & (N-1) & N \end{array}\\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ (N-1) \\ N \end{array} &
\begin{bmatrix}
s+\phi_1+\psi_0 & -\sqrt{\phi_1\psi_1} & 0 & \cdots & \cdot & \cdot & \cdot \\
-\sqrt{\phi_1\psi_1} & s+\phi_2+\psi_1 & -\sqrt{\phi_2\psi_2} & \cdots & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \cdots & -\sqrt{\phi_{N-2}\psi_{N-2}} & s+\phi_{N-1}+\psi_{N-2} & \sqrt{\phi_{N-1}\psi_{N-1}} \\
\cdot & \cdot & \cdot & \cdots & 0 & -\sqrt{\phi_{N-1}\psi_{N-1}} & s+\phi_N+\psi_{N-1}
\end{bmatrix}
\end{array}$$

$|D(s)|$ is a polynominal of degree N in s.

The roots of $|D(s)|$ are the negatives of the eigenvalues of the matrix D(0). The matrix D(0) is a positive definite symmetric tridiagonal matrix. Therefore, it is well known that the eigenvalues are positive and distinct. Hence the roots of the polynominal $|A(s)|$ are real, negative and distinct. Let α_k (k=0,1,2,..N) be the roots of $|A(s)|$ with $\alpha_0=0$. Then

$$|A(s)| = s \prod_{k=1}^{N} (s-\alpha_k)$$

and hence

$$F_n(z) = \frac{|A_{n+1}(s)|}{s \prod_{j=1}^{N} (s-\alpha_j)} \qquad , \; 0 \le n \le N$$

Resolving the right hand side of $F_n(z)$ into partial fractions, replacing s by $(1-z)/z$, using initial conditions and comparing coefficient of z^n, we can find out the value of $F_n(z)$. It will be bounded for $|1+\alpha_k|<1$.

By using QROOT software package, which is developed at Royal Military College, Canada [illegible], M.L. Chaudhry (1992), we can get the roots α_k (k=0,1,2,...,N) of $|A(s)|=0$ called the characteristic equation of P(z). For large N we make the use of IMSL package which require much large memory. Therefore, one might be force to use the main frame computer for greater precision for large N (i.e. N>200).

25 CONCLUSIONS :

We have discussed a discrete time Geom(X)/Geom(Y)/2/N with heterogeneous servers and obtained its transient solution. Numerical Computations can be carried out for several particular cases. The discrete time models are very important for application purposes, it seems this area of research has largely been ignored, particularly when their transient solutions are needed. This work in that sense gives impetus to the analysis of discrete time models with heterogeneous servers.

CHAPTER 3

DISCRETE-TIME TRANSIENT SOLUTION FOR A FIRST PASSAGE TIME DISTRIBUTION IN QUEUEING THEORY

3.1 INTRODUCTION :

This chapter provides transient solution in discrete time for a first-passage time distribution in Queueing Theory under arbitrary initial condition and finite waiting space. Most of the Queueing Theory literature concentrates on finding the steady state solutions or approximations. Very little seems to have been done to evaluate the transient solutions. Even at times steady state solutions are difficult to compute. Chaudhry M.L., Agarwal M. and Templeton J.G.C. (1992) have mostly concentrated on this using the technique of roots. Earlier attempts at finding transient solutions can be [illegible] L. Takacs L. (1962) and Morse P.M. (1958). However, there are computational difficulties with their methods.

Now with the increased skill available in computations with the use of computers, researchers especially in Computer Science have started looking for transient solutions and easy to compute closed form solutions. Recently, Sharma O.P. and Dass J.(1988) have provided transient solutions to a class of Markovian models in Queueing Theory. However, they did not concentrate on the

computational difficulties of finding the roots or eigen values if the matrices involved are large.

However, no attempt seems to have been made to obtain similar results in discrete time for finite waiting space problems in Queueing Theory. As the transient solution is not independent of the initial state of the system it is interesting to know its effect on the system's behavior. Further, some systems may not exist long enough to reach their steady state.

There are several systems which operate at discrete times see Kobayashi H.(1983) As a result, it becomes important to study them. In such cases events are clock controlled.

In this paper we analyze a discrete time Queueing model for the first passage time distribution to a absorbing state given the initial state. Such problems occur not only in Queueing Theory but also in Bio-Science and now in Computer Science. We give closed form solution to this class of problems in terms of the roots of a polynomial in z-transform and results are computed even when the matrices involved are large. The case of 1-channel busy period is also discussed. It is also shown, how the results for the continuous case can be obtained. Interesting analogy exists between the discrete-time models and their continuous-time counterpart. Such an analogy, though simple to prove has

never been shown before.

Results presented in this chapter further unify the treatment given by Chaudhry M.L., Kapur P.K., Templeton J.G.C. (197[illegible]). It is worth noting, though continuous-time models are particular cases of discrete-time models, yet this area of research has remain neglected except some feeble attempts made by few.

3.2 ASSUMPTIONS:

1. The queue consists of finite waiting space.
2. Interarrival and service probabilities are dependent on the state of the system.
3. Interarrival and service time distributions are geometrical and independent of time.
4. Queue discipline is first-in-first-out (FIFO).

3.3 NOTATIONS :

X_k : number of customers in the system at epoch k.

N : size of the queue (maximum).

λ_n : inter-arrival probability when n customers are in the system.

μ_n : service probability when n customers are in the system.

ψ_n : $\lambda_n (1-\mu_n)$.

ϕ_n : $\mu_n (1-\lambda_n)$.

h : absorbing barrier ($h \geq 0$) ($h<n$).

3.4 ANALYSIS OF THE MODEL :

Let X_k be the number of customers in the system at discrete time epoch k. Then X_k, $k \geq 1$ is an integer valued discrete stochastic process taking values $(h, h+1, \ldots, N)$. $X_k = n$ $(h \leq n \leq N)$ implies that there are n customers in the system at discrete time epoch k. When a customer arrives or leaves, a discontinuity in the stochastic process occurs. The process X_k behaves as a discrete-time Markov process and represents the state of the system.

Denote the probability that the system in the state n at the beginning of the m^{th} epoch as $P_m(n)$ $(n = h, h+1, \ldots, N)$.

3.5 BACKWARD FIRST TIME-DISTRIBUTION ANALYSIS :

Considering the case for the elapsed time slot m, the following difference equations may be easily written before the queue lengths for the first time reached h :

$$P_{m+1}(h) = P_m(h) + \phi_{h+1} P_m(h+1) \tag{3.1}$$

$$P_{m+1}(h+1) = P_m(h+1) - (\psi_{h+1} + \phi_{h+1}) P_m(h+1) + \phi_{h+2} P_m(h+2) \tag{3.2}$$

$$P_{m+1}(n) = P_m(n) - (\psi_n + \phi_n) P_m(n) + \psi_{n-1} P_m(n-1) + \phi_{n+1} P_m(n+1)$$
$$(h+2) \leq n \leq (N-1) \tag{3.3}$$

$$P_{m+1}(N) = P_m(N) - \phi_N P_m(N) + \psi_{N-1} P_m(N-1) \tag{3.4}$$

where $\lambda_h = 0$ and $P_0(i) = 1$, $h \leq i \leq N$.

Let $P(n)$ be the steady state distribution, i.e.

$\lim_{m \to \infty} P_m(n) = P(n)$.

If such a distribution exists, it is unique. Solving (5.1) to (5.4) for the stationary case, we get

$P(h) = 1$,

$P(i) = 0$, $\qquad (h+1) \leq i \leq N$

Let $F_z(n)$ be the probability generating function (p.g.f.) of $P_m(n)$ defined as

$G(z, n) = F_z(n) = \sum_{m=0}^{\infty} z^m P_m(n)$, $\qquad |z| \leq 1$.

Taking the p.g.f. of equations (1) to (4), we have

$$AF = \left[\delta_{hh} \ \delta_{h(h+1)} \ \ldots\ldots \ \delta_{hN} \right] \longrightarrow (5.5)$$

Where A is a real tri-diagonal $(N-h+1)\times(N-h+1)$ matrix, F is a column vector and δ_{hi} is the Kronecker delta defined as

$$\delta_{hi} = \begin{cases} 1/z, & i = h \\ 0, & \text{otherwise} \end{cases}$$

Defining $s = (1-z)/z$, we have $A(s) =$

$$\begin{array}{c|c} & \begin{matrix} h & h+1 & h+2 & h+3 & \cdots & N-1 & N \end{matrix} \\ \hline \begin{matrix} h \\ h+1 \\ h+2 \\ \vdots \\ \vdots \\ N-1 \\ N \end{matrix} & \begin{bmatrix} s & \phi_{h+1} & 0 & \cdot & \cdot & \cdot & 0 \\ s & s+\psi_{h+1}+\phi_{h+1} & -\phi_{h+2} & \cdot & \cdot & \cdot & 0 \\ \cdot & -\psi_{h+1} & s+\phi_{h+2}+\psi_{h+2} & -\phi_{h+3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\psi_{N-2} & s+\psi_{N-1}+\phi_{N-1} & & \cdot \\ \cdot & \cdot & \cdot & \cdot & -\psi_{N-1} & & s+\phi_N \end{bmatrix} \end{array}$$

$(N-h+1)\times(N-h+1)$

$$
F = \begin{bmatrix} F_2(h) \\ F_2(h+1) \\ \cdot \\ \cdot \\ F_2(N) \end{bmatrix}_{(N-h+1) \times 1}
$$

From equation (3.5), using Cramer's rule $F_2(n)$ are explicitly determined as

$$
F_2(n) = \frac{|A_{n-h+1}(s)|}{|A(s)|} \qquad h \leq n \leq N
$$

where $A_{n-h+1}(s)$ is obtained from $A(s)$ by replacing the $(n-h+1)^{st}$ column of $A(s)$ by the right hand side of (3.5) and $|A(s)|$ is the determinant of $A(s)$.

Applying some row and column transformations on $|A(s)|$, it may be expressed as $a|D(s)|$, where $D(s)$ is a real, symmetric, tri-diagonal matrix of order $(N-h) \times (N-h)$.

Specifically, D(s) =

$$
\begin{bmatrix}
s+\phi_{h+1} & -\sqrt{\psi_{h+1}\phi_{h+1}} & & & \cdot & \cdot & \cdot & 0 \\
-\sqrt{\psi_{h+1}\phi_{h+1}} & s+\psi_{h+1}+\phi_{h+2} & -\sqrt{\psi_{h+2}\phi_{h+2}} & & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{\psi_{N-2}\phi_{N-2}} & s+\psi_{N-1}+\phi_{N-1} & -\sqrt{\psi_{N-1}\phi_{N-1}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{\psi_{N-1}\phi_{N-1}} & s+\phi_N+\psi_{N-1}
\end{bmatrix}
$$

$|D(s)|$ is a polynomial of degree $(N-h)$ in s. It may be noted that the roots of $|D(s)|$ are the negatives of the eigen values of the matrix $D(0)$.

It may be observed that $D(0)$ is a positive definite, symmetric tri-diagonal matrix. It is well known that its eigen values are positive, real and distinct. Thus, the roots of the polynomial $|A(s)|$ are real, negative and distinct (one root is zero). Let α_k ($k=0,1,2, \ldots, N-h$) be the roots of $|A(s)|$ with $\alpha_0 = 0$. Then,

$$|A(s)| = s\,\pi_{k=1}^{N-h}\,(s-\alpha_k),$$

and hence

$$F_z(n) = \frac{|A_{n-h+1}(s)|}{s\,\pi_{j=1}^{N-h}\,(s-\alpha_j)}, \qquad h \le n \le N$$

Resolving the right hand side of $F_z(n)$ into partial fractions and replacing s by $(1-z)/z$, using initial conditions and comparing the coefficients of z^m, we get

$$F_z(h) = 1 - \pi_{r=h}^{i-1}\,\phi_{r+1}\,\Sigma_{k=1}^{N-h}\,a_{kh}\,(1+\alpha_k)^m$$

$$F_z(n) = \pi_{r=n}^{i-1}\,\phi_{r+1}\,\Sigma_{k=1}^{N-h}\,a_{kn}\,(1+\alpha_k)^m, \qquad h < n < i$$

$$F_z(n) = \Sigma_{k=1}^{N-h}\,a_{kn}\,(1+\alpha_k)^m, \qquad n = i$$

$$P_m(n) = \prod_{i=1}^{n-1} \psi_{i+1} \sum_{k=1}^{N-h} a_{kn} \, (1+\alpha_k)^m , \quad 1 < n \leq N$$

$$a_{kn} = \frac{C_{N-1}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)}$$

$$a_{kn} = \frac{C_{N-1}(\alpha_k)\, D_{n-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)} , \quad h < n < i$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_{i-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)} , \quad n = i$$

$$a_{kn} = \frac{C_{N-n}(\alpha_k)\, D_{i-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)} , \quad i < n \leq N$$

with $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left $(n \times n)$ square matrices formed from $A(s)$ such that

$$|A(s)| = C_{N-h+1}(s) = D_{N-h+1}(s)$$

It may be remarked that for the probabilities $P_m(n)$ $(1 \leq n \leq N)$ to remain bounded, $|1+\alpha_k| < 1$, which is true if $\phi_k \psi_{k+1} < 1$. Under this condition, the sum of the absolute values of the elements in each row of the matrix D(O) is less than 2 and hence from Gerschgorin's theorem $|\alpha_k| < 2$. Hunter J.J.(1983) which implies $|1+\alpha_k| < 1$. $C_n(s)$ and $D_n(s)$ may be determined by the following recurrence relations.

Assuming $C_0(s) = D_0(s) = 1$, $C_1(s)=s+\phi_N$, $D_1(s) = s$ and $\lambda_N=\psi_h=\phi_h=0$.

$$C_i(s) = (s-\psi_{N+1-i}+\phi_{N+1-i})C_{i-1} - \psi_{N+1-i}\phi_{N+2-i}C_{i-2}$$
$$2 \le i \le N-h+1$$

$$D_i(s)= (s-\psi_{h+i-1}-\phi_{h+i-1})D_{i-1} - \psi_{h+i-2}\phi_{h+i-1}D_{i-2}$$
$$2 \le i \le N-h+1$$

Using the standard IMSL package one can find the eigen values and hence the zeros of the polynomial $|A(s)|$. Further using the recurrence relations for $C_i(s)$ and $D_i(s)$, $C_i(s)$ and $D_i(s)$ can easily be evaluated.

Using IMSL to find eigen values for large N would require much larger memory, hence for greater precision for large N (N>200) one might be force to use the Main Frame Computer. To find eigen values or characteristics roots for large N, we make use of IMSL package, which require much large memory. Therefore, one might be force to use the mainframe computer for greater precision for large N. (i.e. N>200). The eigen value or characteristics roots can also be obtained by using QROOT Software Package developed at RMC, Canada by Chaudhary M.L. (1992). Therefore, some comment on QROOT will be in order. The accuracy of the roots by QROOT given by $|P(\alpha_k)| < 10^{-14}$ is not sufficient for the problems under consideration because of the recurrence relations involved in finding the roots and the probabilities [illegible]. However, if it is increased, it takes lot more time

and still the accuracy is not sufficient to meet the requirements. We faced no problems using IMSL as far as accuracy goes. However the problem is with the memory when N is very large. IMSL proves better than GROOT for the problem considered in this paper.

Since $(1+\alpha_k)^m \to 0$ as $m \to \infty$, the steady state distribution $P(n)$ is given by

$$P(h) = 1$$

$$P(n) = 0, \qquad (h+1) \le n \le N$$

See appendix for illustration. Besides, it may be remarked that the solution presented here is expressed as the sum of two parts, one pertaining to the steady state and the other to the transient state.

3.6 IMPORTANT PERFORMANCE MEASURES :

Using closed from expressions for $P_m(n)$, some important measures can be analytically and numerically derived.

1. Expected number of customers in the system (for fixed i)

$$E(X_m) = \Sigma_{n=h}^{N} n \ P_m(n)$$

2. If Y_m denotes the number of customers present in the queue (excluding the customers receiving service)

$$\text{n} \quad E(Y_m) = \Sigma_{n=h+r}^{N} (n-r) \ P_m(n)$$

3. Probability that the system state is greater than a given number c is given by $(c \ge h)$

$$\sum_{n=c}^{N} P_m(n)$$

4. Relaxation time which is a measure of length of time required for the system to settle down to its steady state condition is defined, Morse P.M. (1958), as

$$RT = 1/\min_{i=1}^{N} (-Re(\alpha_i))$$

If $m >> RT$,

$$P_m(n) = P(n), \quad \forall n$$

3.7 GENERAL CASE :

So far initial queue size has been assumed to be fixed and equal to i. It implies that the initial probability vector can contain 1/z in any one place only. We now consider a general case of this problem, where there can be more than one non-zero elements in the initial probability vector. The probability $Q_m(n)$ $(n=h,h+1,\ldots,N)$ (probability of n customers at epoch is irrespective of the state of the system) may be defined as

$$Q_m(n) = \sum_{i=h}^{N} P_m(n,i)\, P_0(i), \quad h \leq n \leq N$$

where $P_m(n,i)$ is $P_m(n)$ for a given i and $P_0(i)$ is the initial i^{th} probability.

3.8 CONTINUOUS TIME CASE :

Letting $\lambda_n = \lambda_n\Delta+O(\Delta)$, $\mu_n = \mu_n\Delta+O(\Delta)$, $m=t$ and $m+1=t+\Delta$ in equations (3.1) to (3.4), the difference equations in m can be transformed to differential equations in t. The transformed equations can be solved for continuous-time probabilities. Alternatively, the roots α_k of the polynomial $|A(s)|$ are transformed to $\alpha_k{}^*\Delta$. It may be noted that $(1+\alpha_k{}^*\Delta)^m$ tends to $e^{\alpha_k{}^*t}$ in continuous time where t is divided into m subintervals of length Δ such that $t = m\Delta$. Treating λ_n and μ_n as interarrival and service rates respectively, one gets the transient solutions for continuous-time model. s may be treated as the transform parameter in the continuous case. Right hand side of (3.5) will have 1 in the i^{th} place.

3.9 FORWARD FIRST PASSAGE TIME :

Next we consider the absorbing barrier on the maximum queue size, we may write the difference equations as

$$P_{m+1}(0) - P_m(0) = -\psi_0\, P_m(0) + \phi_1\, P_m(1)$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n + \phi_n)P_m(n) + \psi_{n-1}P_m(n-1) + \phi_{n+1}P_m(n+1)$$

$$1 \le n \le N-2$$

$$P_{m+1}(N-1) - P_m(N-1) = -(\psi_{N-1} + \phi_{N-1})P_m(N-1) + \psi_{N-2}P_m(N-2) \quad P_{m+1}(N)$$

$$P_{m+1}(N) - P_m(N) = \psi_{N-1}\, P_m(N-1)$$

where $\mu_0 = 0$ and $P_0(i) = 1$, $\quad 0 \le i \le N$

Proceeding as above the steady state probabilities may be given as

$$P(i) = 0 \qquad 0 \le i \le N-1$$

$$P(N) = 1$$

The probabilities $P_m(n)$ may be expressed as

$$P_m(n) = \pi_{r=n}^{i-1} \phi_{r+1} \Sigma_{k=1}^{N} a_{kn} (1+\alpha_k)^m \quad , \quad 0 \le n \le i$$

$$P_m(n) = \Sigma_{k=1}^{N} a_{kn} (1+\alpha_k)^m \quad , \quad n = i$$

$$P_m(n) = \pi_{r=i}^{n-1} \psi_r \Sigma_{k=1}^{N} a_{kn} (1+\alpha_k)^m \quad , \quad i < n \le N$$

$$P_m(N) = 1 + \pi_{r=i}^{N-1} \psi_r \Sigma_{k=1}^{N} a_{kN} (1+\alpha_k)^m$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k\, \pi_{j=1 \ne k}^{N} (\alpha_k - \alpha_j)} \quad , \quad 0 \le n \le i$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k\, \pi_{j=1 \ne k}^{N} (\alpha_k - \alpha_j)} \quad , \quad n = i$$

$$a_{kn} = \frac{C_{N-n}(\alpha_k)\, D_i(\alpha_k)}{\alpha_k\, \pi_{j=1 \ne k}^{N} (\alpha_k - \alpha_j)} \quad , \quad i < n < N$$

$$a_{kN} = \frac{D_i(\alpha_k)}{\alpha_k\, \pi_{j=1 \ne k}^{N} (\alpha_k - \alpha_j)}$$

$C_n(s)$ and $D_n(s)$ are as defined earlier.

3.10 i-CHANNEL BUSY PERIOD :

We define i-channel busy period $(0<i\le N)$ to begin with an arrival to the system at an epoch when there are (i-1) customers in the system to the very next epoch when there are again (i-1) customers in the system. Assuming λ_n and μ_n to be the interarrival and service probabilities respectively, when there are n customers in the system, the following difference equations may be writven

$$P_{m+1}(i-1) - P_m(i-1) = \phi_i P_m(i)$$

$$P_{m+1}(i) - P_m(i) = -(\psi_i+\phi_i)P_m(i) + \phi_{i+1}P_m(i+1)$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n+\phi_n)P_m(n) + \psi_{n-1}P_m(n-1) + \phi_{n+1}P_m(n+1)$$

$$i \le n \le (N-1)$$

$$P_{m+1}(N) - P_m(N) = -\phi_N P_m(N) + \psi_{N-1}P_m(N-1)$$

where $\lambda_N = 0$ and $P_0(i) = 1$

The solution to these equations can be obtained as before with h=i-1 and h+1=i. It may be noted that $P_m(i-1)$ and $\phi_i P_m(i)$ are respectively the probability distribution and probability mass function of the busy period.

3.11 NUMERICAL RESULTS :

We give below the numerical results for both the discrete and continuous cases for each of the models discussed above. For the sake of convenience, results for

only moderate values of N are given though there were no problems even for large values of N.

Case (i) **Backward First Passage Time**

Discrete case

Assume $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=4,5,6,\ldots,20$, $h=4$ $(h\leq i\leq N)$ and $P_0(i) = 1/17$. Table 3.1 gives the probabilities $P_m(n)$ for different i $(i=4,5,\ldots,20)$, the unconditional probabilities $Q_m(n)$ and the steady state probabilities $P(n)$. The last two rows give the values of $E(X_m)$ and $E(Y_m)$. For $i=20$, the time to reach steady state is $m = 700$ which is $\gg RT = 47$.

Continuous Case

Assume $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $t=10$, $i=4,5,6,\ldots,20$, $h=4$ $(h\leq i\leq N)$ and $P_0(i) = 1/17$. Table 3.2 gives the probabilities $P_n(t)$ for different i $(i=4,5,\ldots,20)$, the unconditional probabilities $Q_n(t)$ and the steady state probabilities $p(n)$. The last two rows give values of $E(X_m)$ and $E(Y_m)$. For $i=20$, the time to reach steady state is $t = 900$ which is $\gg RT = 86$.

Case (ii) **Forward First Passage Time**

Discrete Case

Assume $r=6$, $N=16$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=0,1,2,\ldots,16$ and $P_0(i)=1/17$. Table 3.3 gives the probabilities $P_m(n)$ for different i $(i=4,5,\ldots,20)$, the unconditional probabilities $Q_m(n)$ and the steady state

probabilities $P(n)$. The last two rows give values of $E(X_m)$ and $E(Y_m)$.

Continuous Case

Assume $r=6$, $N=16$, $\lambda=0.8$, $\mu=0.15$, $t=10$, $i=0,1,2,\ldots,16$ and $P_0(i)=1/17$. Table 3.4 gives the probabilities $P_n(t)$ for different i ($i=4,5,\ldots,20$), the unconditional probabilities $Q_n(t)$ and the steady state probabilities $P(n)$. The last two rows give values of $E(X_m)$ and $E(Y_m)$.

Case (iii) **i-channel busy period**

Assume $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=5,6$. Table 3.5 gives the probabilities $P_m(n)$ and the steady state probabilities.

3.12 CONCLUSIONS :

We have discussed a discrete-time Markovian model Geom(n)/Geom(n)/r/N for a first passage problem and obtained its transient solution. Numerical computations have been carried out extensively for the backward and forward first passage models and also for i-channel busy period which is a particular case of backward first passage time model. Computations have also been carried out for their counterpart in the continuous time. The discrete-time models are very important for application purposes such as Computers, it seems this area of research has largely been ignored

particularly when the transient solutions are needed. This study in that sense gives impetus to the analysis of discrete time models. An analogy is also established between the discrete and continuous time models. Such an analogy has not been illustrated before. Finally the accuracy of the eigen values/roots that is needed in the kind of problems under study is very very large because of the recurrence relations involved in computing the probabilities.

Table 3.1 : Probabilities $P_m(n)$, $P(n)$ and means for $Geom(n)/Geom(n)/r/N$ with $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=4,5,6,\ldots,20$, $h=4$ $(h \le i \le N)$' and $Q_m(n)$ with $P_0(i) = 1/17$

n \ i	4	5	6	18	19	20	$Q_m(n)$	P(n)
4	1.0000	0.5507	0.3163	0.0000	0.0000	0.0000	0.1217	1.0000
5	0.0000	0.1367	0.1874	0.0000	0.0000	0.0000	0.0370	0.0000
6	0.0000	0.2083	0.3022	0.0000	0.0000	0.0000	0.0717	0.0000
7	0.0000	0.0798	0.1376	0.0000	0.0000	0.0000	0.0627	0.0000
8	0.0000	0.0204	0.0444	0.0000	0.0000	0.0000	0.0597	0.0000
9	0.0000	0.0036	0.0102	0.0000	0.0000	0.0000	0.0590	0.0000
10	0.0000	0.0005	0.0017	0.0000	0.0000	0.0000	0.0589	0.0000
11	0.0000	0.0000	0.0002	0.0003	0.0000	0.0000	0.0588	0.0000
12	0.0000	0.0000	0.0000	0.0022	0.0003	0.0000	0.0588	0.0000
13	0.0000	0.0000	0.0000	0.0115	0.0022	0.0003	0.0588	0.0000
14	0.0000	0.0000	0.0000	0.0421	0.0115	0.0023	0.0588	0.0000
15	0.0000	0.0000	0.0000	0.1038	0.0421	0.0124	0.0587	0.0000
16	0.0000	0.0000	0.0000	0.2020	0.1103	0.0466	0.0583	0.0000
17	0.0000	0.0000	0.0000	0.2551	0.2039	0.1254	0.0566	0.0000
18	0.0000	0.0000	0.0000	0.2134	0.2619	0.2390	0.0517	0.0000
19	0.0000	0.0000	0.0000	0.1164	0.2290	0.3124	0.0417	0.0000
20	0.0000	0.0000	0.0000	0.0472	0.1388	0.2616	0.0271	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_m)$	4.0000	4.8954	5.4449	16.9809	17.8967	18.5685	11.1196	4.0000
$E(Y_m)$	0.0000	0.1335	0.2648	10.9809	11.8967	12.5685	5.3999	0.0000

Table 3.2 : Probabilities $P_n(t)$, $P(n)$ and means for M/M/r?N with $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $t=10$, $i=4,5,6,\dots,20$, $h=4$ $(h \le i \le N)$ and $Q_n(t)$ with $P_0(i) = 1/17$

n \ t	4	5	6	18	19	20	$Q_n(t)$	P(n)
4	1.0000	0.8219	0.6693	0.0018	0.0009	0.0005	0.2459	1.0000
5	0.0000	0.0153	0.0269	0.0013	0.0007	0.0005	0.0169	0.0000
6	0.0000	0.0239	0.0422	0.0029	0.0016	0.0011	0.0285	0.0000
7	0.0000	0.0279	0.0498	0.0053	0.0032	0.0021	0.0377	0.0000
8	0.0000	0.0277	0.0502	0.0092	0.0058	0.0041	0.0448	0.0000
9	0.0000	0.0244	0.0452	0.0151	0.0101	0.0075	0.0498	0.0000
10	0.0000	0.0197	0.0371	0.0235	0.0167	0.0131	0.0532	0.0000
11	0.0000	0.0145	0.0282	0.0349	0.0262	0.0216	0.0553	0.0000
12	0.0000	0.0100	0.0199	0.0490	0.0392	0.0338	0.0564	0.0000
13	0.0000	0.0064	0.0131	0.0654	0.0557	0.0501	0.0568	0.0000
14	0.0000	0.0038	0.0081	0.0828	0.0750	0.0702	0.0565	0.0000
15	0.0000	0.0022	0.0048	0.1995	0.0957	0.0929	0.0556	0.0000
16	0.0000	0.0012	0.0026	0.1137	0.1155	0.1159	0.0541	0.0000
17	0.0000	0.0006	0.0014	0.1236	0.1317	0.1358	0.0520	0.0000
18	0.0000	0.0003	0.0007	0.1279	0.1416	0.1494	0.0492	0.0000
19	0.0000	0.0001	0.0003	0.1260	0.1437	0.1538	0.0457	0.0000
20	0.0000	0.0001	0.0002	0.1181	0.1367	0.1476	0.0416	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_t)$	4.0000	4.8287	5.5862	15.9648	16.4125	16.6546	10.8242	4.0000
$E(Y_t)$	0.0000	0.4679	0.9517	9.9698	10.4150	10.6561	5.3330	0.0000

Table 3.3 : Probabilities $P_m(n)$, $P(n)$ and means for $Geom(n)/Geom(n)/r/N$ with $r=6$, $N=16$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=0,1,2,\ldots,16$, and $Q_m(n)$ with $P_0(i) = 1/17$

n \ i	0	1	2	14	15	16	$Q_m(n)$	$P(n)$
0	0.0004	0.0003	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000
1	0.0075	0.0051	0.0033	0.0000	0.0000	0.0000	0.0012	0.0000
2	0.0510	0.0377	0.0269	0.0000	0.0000	0.0000	0.0093	0.0000
3	0.1763	0.1440	0.1144	0.0000	0.0000	0.0000	0.0399	0.0000
4	0.3255	0.3006	0.2707	0.0000	0.0000	0.0000	0.1014	0.0000
5	0.3047	0.3290	0.3448	0.0000	0.0000	0.0000	0.1567	0.0000
6	0.1219	0.1611	0.2034	0.0000	0.0000	0.0000	0.0427	0.0000
7	0.0120	0.0204	0.0324	0.0003	0.0000	0.0000	0.0753	0.0000
8	0.0007	0.0017	0.0036	0.0022	0.0003	0.0000	0.0614	0.0000
9	0.0000	0.0001	0.0003	0.0115	0.0022	0.0000	0.0591	0.0000
10	0.0000	0.0000	0.0000	0.0421	0.0114	0.0000	0.0586	0.0000
11	0.0000	0.0000	0.0000	0.1097	0.0411	0.0000	0.0579	0.0000
12	0.0000	0.0000	0.0000	0.2015	0.1046	0.0000	0.0552	0.0000
13	0.0000	0.0000	0.0000	0.2527	0.1832	0.0000	0.0478	0.0000
14	0.0000	0.0000	0.0000	0.2042	0.2062	0.0000	0.0338	0.0000
15	0.0000	0.0000	0.0000	0.1916	0.1228	0.0000	0.0154	0.0000
16	0.0000	0.0000	0.0000	0.0842	0.3282	1.0000	0.0842	1.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_m)$	4.2848	4.4840	4.6854	13.0336	14.2047	16.0000	8.2938	16.0000
$E(Y_m)$	0.1481	0.2075	0.2799	8.0336	9.2047	11.0000	3.5030	11.0000

Table 3.4 : Probabilities $P_n(t)$, $P(n)$ and means for $M/M/r/N$ with $r=6$, $N=16$, $\lambda=0.8$, $\mu=0.15$, $t=10$, $i=0,1,2,\ldots,16$, and $Q_n(t)$ with $P_0(i) = 1/17$

n \ i	0	1	2	14	15	16	$Q_n(t)$	P(n)
0	0.0159	0.0123	0.0095	0.0000	0.0000	0.0000	0.0038	0.0000
1	0.0057	0.0545	0.0449	0.0003	0.0001	0.0000	0.0184	0.0000
2	0.1358	0.1199	0.1048	0.0015	0.0006	0.0000	0.0454	0.0000
3	0.1869	0.1747	0.1613	0.0043	0.0019	0.0000	0.0757	0.0000
4	0.1920	0.1893	0.1839	0.0010	0.0042	0.0000	0.0958	0.0000
5	0.1562	0.1619	0.1650	0.0153	0.0072	0.0000	0.0982	0.0000
6	0.1034	0.1123	0.1199	0.0212	0.0104	0.0000	0.0849	0.0000
7	0.0643	0.0735	0.0824	0.0294	0.0148	0.0000	0.0741	0.0000
8	0.0377	0.0454	0.0537	0.0389	0.0201	0.0000	0.0653	0.0000
9	0.0209	0.0266	0.0332	0.0484	0.0257	0.0000	0.0578	0.0000
10	0.0110	0.0147	0.0194	0.0561	0.0305	0.0000	0.0507	0.0000
11	0.0055	0.0078	0.0108	0.0600	0.0333	0.0000	0.0435	0.0000
12	0.0026	0.0039	0.0057	0.0583	0.0329	0.0000	0.0356	0.0000
13	0.0012	0.0018	0.0029	0.0502	0.0287	0.0000	0.0271	0.0000
14	0.0005	0.0008	0.0013	0.0364	0.0210	0.0000	0.0180	0.0000
15	0.0002	0.0002	0.0006	0.0186	0.0108	0.0000	0.0088	0.0000
16	0.0002	0.0004	0.0007	0.5520	0.7578	1.0000	0.1909	1.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_t)$	4.2378	4.5005	4.7793	13.4019	14.6366	16.0000	8.5799	16.0000
$E(Y_t)$	0.5530	0.6783	0.8278	8.4257	9.6472	11.0000	4.0560	11.0000

Table 3.5 : Probabilities $P_m(n)$, $P(n)$ for Geom(n)/Geom(n)/r/n for i-channel busy period with $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=5,6$.

$i=5$

n	$P_m(n)$	$P(n)$
4	0.5507	1.0000
5	0.1367	0.0000
6	0.2083	0.0000
7	0.0798	0.0000
8	0.0204	0.0000
9	0.0036	0.0000
10	0.0005	0.0000
11	0.0000	0.0000
12	0.0000	0.0000
13	0.0000	0.0000
14	0.0000	0.0000
15	0.0000	0.0000
16	0.0000	0.0000
17	0.0000	0.0000
18	0.0000	0.0000
19	0.0000	0.0000
20	0.0000	0.0000

$i=6$

n	$P_m(n)$	$P(n)$
5	0.8524	1.0000
6	0.0108	0.0000
7	0.0186	0.0000
8	0.0227	0.0000
9	0.0231	0.0000
10	0.0208	0.0000
11	0.0169	0.0000
12	0.0127	0.0000
13	0.0088	0.0000
14	0.0057	0.0000
15	0.0035	0.0000
16	0.0020	0.0000
17	0.0010	0.0000
18	0.0005	0.0000
19	0.0003	0.0000
20	0.0002	0.0000

CHAPTER FOUR

NUMERICAL COMPUTATIONS OF DISCRETE-TIME SOLUTIONS FOR A MULTI-SERVER QUEUE WITH BALKING AND RENEGING

4.1 INTRODUCTION :

Queueing Theory litrature mostly concentrates on finding the steady state solutions or approximations. Little seems to have been done to evaluate the transient solutions. Even at times steady state solutions are difficult to compute. Chaudhary M.L. (1991) have mostly concentrated on this. Problem in principle has been to find the roots of a polynominal in s (laplace transform variable). Earlier attempts at finding the transient solutions can be attributed to Takacs L. (1952) and Morse P.M. (1958). However there are difficulties in computations with these methods. Recently Sharma O.P. (1990) have provided transient solutions to a class of Markovian models in queueing theory. However, he did not look into the computational difficulties involved if the matrices are large.

Moreover, no attempt seems to have been made to obtain similar results in discrete time for finite waiting space problems in queueing theory. As the transient solutions

are not independent at the initial state of the systems, it is interesting to know its effect on system's behaviour. Further, some systems may not exists long enough to reach steady state.

There are several systems, which operate at discrete times, see Kobayashi H. (1983). Therefore, it becomes important to study them. In such cases, events are clock controlled.

In this chapter we analyze a discrete time multi-server queue with balking and reneging given the initial state. We also discuss the case when the initial state is arbitrary. We give closed from solutions to this class of problems in terms of roots of a polynominal in z-transform and results are computed even when the matrices involved are large. It is also shown, how the results in the continious case can be obtained. Interesting analogy exists between the discrete time models and their continous time counterparts. Such as analogy, though simple in nature has not been shown before. Results presented in this chapter further unify the treatment given earlier in [1-2,6].

It is worth nothing, though continous time models are particular cases of discrete time models, yet this area of research has remain neglected. It is in this sense that

this chapter should simulate the study of discrete time models in other areas such as computer science. Finally extensive numerical computations were performed in order to judge the accuracy of the results (see comments on computational aspects). Case of Machine Interference problems is also given.

4.2 ASSUMPTIONS :

1. The queue size is finite.
2. Inter-arrival and service probabilities "are dependent on the state of the system.
3. Inter-arrival and service time distributions are geometric but independent of time.
4. Queue discipline is First Come First Serve (FCFS).

4.3 NOTATIONS :

X_k : Number of customers in the queue at time epoch k

N : Maximum queue size.

ϵ_n : Inter-arrival probability when n customers are in the system.

μ_n : Service probability when n customers are in the system.

T_n : $\epsilon_n(1-\mu_n)$

θ_n : $\mu_n(1-\epsilon_n)$

4.4 MODEL ANALYSIS :

We develop a general discrete time Markov model for a finite waiting space queueing system and analyze the effects of customer impatience on its transient behaviour. Impatience can be due to balking, reneging or both. Balking is the reluctance of a customer to join the queue upon arrival. Reneging is the reluctance of a customer to remain in the queue after joining it and leaving the queue without being serviced. It may be noted that initial number of customers c will not renege because of their immediate entry to the service facility. Still these c customers join the queue with some balking probability. We assume that inter-arrival and service times have geometric distributions with parameters ϵ and μ repectively. An arriving customer balks with probability n/N n=0,1,2,....N. Thus inter-arrival probability may be defined as

$$\epsilon_n = \epsilon(1-n/N) \qquad 0 \le n \le N$$

A customer may renege after joining the queue if he or she decides the certain waiting time will be larger that can be tolerated. This reneging time is assumed to have a geometric distribution with parameters Ω. Since any one of the (n-c) customers may renege, the reneging probability may be expressed as

$$\begin{array}{ll} 0 & \text{for } 0 \le n \le c-1 \\ (n-c)\Omega & \text{for } c \le n \le N \end{array}$$

Thus the service probability may be expressed as

$$\mu_n = \begin{cases} n\mu, & \text{for } 0 \le n \le c-1 \\ c\mu+(n-c)\Omega & \text{for } n \le c \le N \end{cases}$$

Let X_k be the number of customers in the system at discrete time epoach k. Then X_k, $k \ge 0$ is an integer valued discrete stochastic process taking values 0,1,2,......N. $X_k = n$ $(0 \le n \le N)$ implies that there are n customers in the system at epoch k. As and when a customers arrives or leaves, a discontinuity in the stochastic process occurs. Thus the process X_k behaves as a discrete-time Markov process and represents the state of the system.

Denote the probability that the system is in state n at the m^{th} epoch as $p_m(n)$ $(0 \le n \le N)$. The following difference equations may easily be written

$$p_{m+1}(0) = p_m(0)(1-\epsilon) + p_m(1)\mu(1-(N-1)\epsilon/N) \longrightarrow (4.1)$$

$$p_{m+1}(n) = p_m(n)(1-(N-n)\epsilon/N-n\mu+2n\epsilon\mu(N-n)/N) + p_m(n-1)(1-(n-1)\mu)$$
$$(N-n+1)\epsilon/N+p_m(n+1)(n+1)\mu(1-(N-n-1)\epsilon/N),\quad 1\le n\le c-1 \longrightarrow (4.2)$$

$$p_{m+1}(c) = p_m(c)(1-(N-c)\epsilon/N-c\mu+2c\epsilon\mu(N-c)/N) + p_m(c-1)(1-(c-1)\mu)$$
$$(N-c+1)\epsilon/N+p_m(c+1)(c\mu+\Omega)(1-(N-c-1)\epsilon/N) \longrightarrow (4.3)$$

$$p_{m+1}(n) = p_m(n)(1-(N-n)\epsilon/N-c\mu-(n-c)\Omega+2(c\mu+(n-c)\Omega)\epsilon(N-n)/N)+ p_m(n-1)(1-c\mu-(n-c-1)\Omega)(N-n+1)\epsilon/N+p_m(n+1) (c\mu+(n-c+1)\Omega)(1-(N-n-1)\epsilon/N), \quad c+1\le n\le N-1 \longrightarrow (4.4)$$

$$p_{m+1}(N)=p_m(N)(1-c\mu-(N-c)\Omega)+p_m(N-1)(1-c\mu-(N-c-1)\Omega)\epsilon/N \longrightarrow (4.5)$$

with $p_o(i) = 1, \quad 0 \le i \le N$

Let $p_z(n)$ be the p.g.f. of $p_m(n)$ defined as

$$p_z(n) = \sum_{m=0}^{\infty} z^m \; p_m(n), \qquad |z| \le 1$$

Taking p.g.f. of equations (4.1) to (4.5), we get

$$Ap= \left[\delta_{k0}, \delta_{k1}, \ldots\ldots\ldots\ldots, \delta_{kN} \right]' \longrightarrow (4.6)$$

where A is a (N+1)x(N+1) tridiagonal matrix with real coefficients. p is a (N+1)x1 column vector and δ_{ki} is the Kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z, & k=i \\ 0 & \text{otherwise} \end{cases}$$

Defining $s = (1-z)/z$, we have

$$A(s) = \begin{bmatrix} s+\tau_0 & -\phi_1 & 0 & \cdots & 0 & 0 & 0 \\ -\tau_0 & s+\phi_1+\tau_1 & -\phi_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tau_{N-2} & s+\tau_{N-1}+\phi_{N-1} & -\phi_N \\ 0 & 0 & 0 & \cdots & & -\tau_{N-1} & s+\phi_N \end{bmatrix}$$

where τ_i and ϕ_i are defined above, and

$$p = \begin{bmatrix} p_z(0) \\ p_z(n) \\ \vdots \\ p_z(N-1) \\ p_z(N) \end{bmatrix}$$

From equation (4.6), using Cramer's rule, we may determine $p_z(n)$ explicitely as

$$p_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|}, \qquad 0 \le n \le N$$

where $A_{n+1}(s)$ is obtained from $A(s)$ by replacing the $(n+1)^{th}$ column of $A(s)$ by the right hand side in (4.6) and $|A(s)|$ is the determinant of $A(s)$.

We may observe that $|A(s)| = sg_N(s)$, where $g_n(s)$ satisfies the recurrence relation

$$g_n(s) - (s+\tau_{N-n}+\phi_{N-n+1})g_{n-1}(s) + \tau_{N-n+1}\phi_{N-n+1}g_{n-2}(s) = 0, \qquad 1 \le n \le N$$

with $g_{-1}(s) = 0$ and $g_0(s) = 1$.

$g_N(s)$ may also be expressed as the determinant of NxN real symmetric matrix g(s) as

$$
g(s) = \begin{bmatrix}
s+\phi_1+T_0 & -\sqrt{\phi_1 T_1} & 0 & \cdot & \cdot & \cdot \\
-\sqrt{\phi_1 T_1} & s+\phi_2+T_1 & -\sqrt{\phi_2 T_2} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\sqrt{\phi_{N-2} T_{N-2}} & \cdot & s+\phi_{N-1}+T_{N-2} & -\sqrt{\phi_{N-1} T_{N-1}} \\
\cdot & \cdot & 0 & \cdot & -\sqrt{\phi_{N-1} T_{N-1}} & s+\phi_N+T_{n-1}
\end{bmatrix}
$$

The zeros of $g_N(s)$ are the negatives of the eigen values of the matrix g(0). g(0) is a positive definite symmetric tri-diagonal matrix. Hence its eigen values are real, positive (>0) and distinct. Hence the roots of $g_N(s)$ are real, negative and distinct. Let $\alpha_1, \alpha_2, \ldots \alpha_N$ be the roots of $g_N(s)$. Thus,

$$
p_z(n) = \frac{|A_{n+1}(s)|}{s \prod_{i=1}^{N} (s-\alpha_i)} \qquad 0 \le n \le N
$$

Resolving the right hand side of $p_z(n)$ into partial fractions, replacing s by $(1-z)/z$ and comparing the coefficients of z^m we have

Case I

for $0 \leq i \leq c-1$

$$b_n + \frac{i!}{n!}\,\mu^{i-n}\;X_n \sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m, \qquad 0 \leq n \leq i$$

$$p_m(n) = b_n + \sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m, \qquad n = i$$

$$b_n + \frac{(n-i)!}{(N-n)!}\,(\epsilon/N)^{n-i} \prod_{j=i}^{n-1} (1-j\mu) \sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m,$$

$$i < n \leq c$$

$$b_n + \frac{(n-i)!}{(N-n)!}\,(\epsilon/N)^{n-i} \prod_{j=i}^{c-1} (1-j\mu) \prod_{k=c}^{n-1} (1-(c\mu+(k-c)\Omega)).$$

$$\sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m, \qquad c < n \leq N$$

where

$$X_n = \prod_{j=n}^{i-1} (1-\epsilon+(j+1)\epsilon/N)$$

Case II

For $c \leq i \leq N$

$$b_n + \frac{(c-1)!}{n!}\,\mu^{c-n-1} \prod_{k=1}^{i-c+1} (c\mu+(i-c+1-k)\Omega).$$

$$X_n \sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m, \qquad 0 \leq n \leq c-1$$

$$b_n + \prod_{k=1}^{i-n} (c\mu+(i-c+1-k)\Omega)\;X_n \sum_{k=1}^{n} a_{kn}\,(1-\alpha_k)^m, \qquad c \leq n < i$$

$$p_m(n) =$$

$$b_n + \sum_{k=1}^{N} a_{kn}(1-\alpha_k)^m, \qquad n = i$$

$$b_n + \frac{(n-i)!}{(N-n)!}(\epsilon/N)^{n-1}\prod_{k=i}^{n-1}(1-(c\mu+(k-c)\Omega)).$$
$$\sum_{k=1}^{N} a_{kn}(1-\alpha_k)^m, \qquad i+1 \le n \le N$$

where a_{kn}'s are defined as

$$a_{kn} = \begin{cases} \dfrac{C_{N-1}(\alpha_k)D_n(\alpha_k)}{\alpha_k \prod_{\substack{j=1 \\ \ne k}}^{N}(\alpha_k-\alpha_j)}, & 0 \le n \le i \\[2ex] \dfrac{C_{N-n}(\alpha_k)D_1(\alpha_k)}{\alpha_k \prod_{\substack{j=1 \\ \ne k}}^{N}(\alpha_k-\alpha_j)}, & i < n \le N \end{cases}$$

with $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left (nxn) square matrices formed from A(s) such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

$C_n(s)$ and $D_n(s)$ may be determined by the following recursive relations

$$C_n(s) = (s+\varnothing_{N-n+1}+T_{N-n+1})C_{n-1}(s)-T_{N-n+1}\varnothing_{N-n+2}C_{n-2}(s),$$
$$1 \le n \le N+1$$

with $C_0(s) = 1$ and $C_{-1}(s) = 0$

$$D_n(s) = (s+\emptyset_{n-1}+\tau_{n-1})D_{n-1}(s)-\emptyset_{n-1}\tau_{n-2}D_{n-2}(s),$$

$$1 \le n \le N+1$$

with $D_0(s) = 1$ and $D_{-1}(s) = 0$

and

$$p_0^{-1} = \begin{cases} 1 + \sum_{r=1}^{N} N_{C_r} (\epsilon/N\mu)^r \dfrac{\prod_{i=1}^{r}(1-(i-1)\mu)}{\prod_{i=1}^{r}(1-\epsilon+i\epsilon/N)}, & c = N \\[2ex] 1 + \sum_{r=1}^{c} N_{C_r} (\epsilon/N\mu)^r \dfrac{\prod_{i=1}^{r}(1-(i-1)\mu)}{\prod_{i=1}^{r}(1-\epsilon+i\epsilon/N)} + \\ \sum_{r=c+1}^{N} \dfrac{N!\epsilon^r \prod_{i=1}^{c}(1-(i-1)\mu) \prod_{i=c+1}^{r}(1-(c\mu+(i-c-1)\Omega))}{N^r (N-r)!\, c!\, \mu^c \prod_{i=c+1}^{r}(c\mu+(i-c)\Omega) \prod_{i=1}^{r}(1-\epsilon-i\epsilon/N)}, & c < N \end{cases}$$

$$p_r p_0^{-1} = \begin{cases} N_{C_r} (\epsilon/N\mu)^r \dfrac{\prod_{i=1}^{r}(1-(i-1)\mu)}{\prod_{i=1}^{r}(1-\epsilon+i\epsilon/N)}, & 1 \le r \le c \\[2ex] \dfrac{N!\epsilon^r \prod_{i=1}^{r}(1-(i-1)\mu) \prod_{i=c+1}^{r}(1-(c\mu+(i-c-1)\Omega)}{N^r (N-r)!\, c!\, \mu^c \prod_{i=c+1}^{r}(c\mu+(i-c)\Omega) \prod_{i=1}^{r}(1-\epsilon+i\epsilon/N)}, & c+1 \le r \le N \end{cases}$$

It may be remarked that for probabilities $p_m(n)$ to remain bounded $|1+\alpha_k| < 1$, which is true if $\emptyset_k+\tau_{k-1} < 1$. Under this condition, the sum of the absolute values of the

elements in each row of the matrix D(O) is less than 2 and hence from the Gerschgorin's theorem $|\alpha_k| < 2$ (see Hunter J.J. (1983)), which implies $|1+\alpha_k| < 1$.

Using IMSL package, we can get the eigen values(roots) of $g(O)(g_N(s))$. The routines of this package are quite efficient and produce results to a high degree of accuracy, even when the matrix size is large (>50).

Since $(1+\alpha_k)^m \to 0$ as $m \to \infty$, the steady state distribution of $p_m(n)$ may be defined as

$$p(n) = \lim_{m \to \infty} p_m(n) = b_n \qquad 0 \le n \le N$$

It may be noted that the values of $p_m(n)$ have been expressed as the sum of two expressions, one pertaining to the steady state and the other pertaining to the transient state.

Important Performance Measures :

Using explicit expressions for $p_m(n)$, some important measures can be defined as under (for fixed i)

1. Mean number of customers in the system at epoch m

$$E(X_m) = \sum_{n=0}^{N} n\, p_m(n)$$

2. Mean number of customers in the queue (excluding those in service) at epoch m

$$E(Y_m) = \sum_{n=c}^{N} (n-c)\, p_m(n)$$

3. Probability there are r or more customers in the system at epoch m

$$\sum_{n=r}^{N} p_m(n)$$

4. Probability all servers are busy at epoch m

$$E(Z_m) = \sum_{n=c}^{N} p_m(n)$$

5. Relaxation Time (RT) (a measure of the length of time required by the system to settle to its steady state [Morse P.M. (1958)]) may be defined as

$$RT = \frac{1}{\min\limits_{i=1}^{N}(-\alpha_i)}$$

If $m \gg RT$ then $p_m(n) \approx p(n)$

6. The probability of balking at epoch m

$$E(A_m) = \sum_{n=0}^{N} (n/N)\ p_m(n)$$

7. The probability of waiting up to epoch m in the queue by those joining it

$$E(B_m) = \sum_{n=c+1}^{N} (1-n/N)\ p_m(n)$$

4.5 GENERAL CASE :

So far we have assumed that the initial queue size is fixed and equal to i *i.e.* 1/z occurs in only one position in the initial probability vector. This assumption is important when we are interested in the transient solution.

The steady state solution does not depend on the initial probability vector. We now consider a more general case of this problem.

When there are more than one non-zero elements in the initial probability vector. The probability $Q_m(n)$ $(n=0,1,\ldots,N)$ defined as the probability of n customers in the system at epoch m irrespective of the state of the system may be defined as

$$Q_m(n) = \sum_{i=0}^{N} p_m(n,i)p_o(i), \qquad 0 \le n \le N$$

where $p_o(i)$ is the i^{th} element of the initial probability vector and $p_m(n,i)$ is the probability of n customers in the system at epoch m assuming i as the initial number of customers.

4.6 CONTINUOUS TIME CASE :

Letting $\epsilon_n = \epsilon_n\theta + O(\theta)$, $\mu_n = \mu_n\theta + O(\theta)$, m=t and m+1=t+$\theta$ in equations (4.1) to (4.5), the difference equations in m can be transformed into differential equations in t. We can then proceed as above to get continuous-time solution from the transformed equations. Alternatively, the root equation can be changed to get the continuous time solution from the final discrete time solution. The roots of α_k of $|A(s)|=0$ are transformed to α_k'. It is easy to see that

$(1+\alpha_k"\theta)^m$ tends to e_k^m "t in continuous time when t is divided into m sub-intervals each of length θ such as $t=m\theta$. Moreover, parameter s itself may be treated as the transform parameter in continuous time. Right hand side of (4.6) will have 1 in the i^{th} place instead of 1/z.

Treating $\Omega=0, \epsilon_i=N\epsilon$, $(i=0,1,2,\ldots\ldots N)$, b_i's represent the steady-state probabilities for a geom(n)/geom(n)/c/N machine interference model.

4.7 NUMERICAL RESULTS :

We give below the numerical results for both the discrete and continuous cases for each of the models disscussed above.

Case (1) Balking and Reneging

Discrete Case

Assume C = 5, N = 20, $\epsilon = 0.5$, $\mu = 0.15$, m = 10 $\Omega=0.005$, $\epsilon_i=(1-i/N)\epsilon$, $\mu_i=i\mu$, for $0 \le i \le c$, $\mu_i=c\mu+(i-c)\Omega$ for $c < i \le N$ and $p_o(i) = 1/21$. Table 1 gives the probabilities $p_m(n)$ for different $i(0 \le i \le 20)$, the unconditional probabilities $Q_m(n)$ and the steady-state probabilities p(n). The last five rows give the values for $E(X_m)$, $E(Y_m)$, $E(Z_m)$, $E(A_m)$ and $E(B_m)$. The epoch to reach steady state is $m \approx 300$ which is $\gg$ RT = 32.

$c < i \le N$ and $p_o(i) = 1/21$. Table 2 gives the probabilities $p_t(n)$ for different $i(0 \le i \le 20)$, the unconditional probabilities $Q_t(n)$ and the steady-state probabilities $p(n)$. The last five rows give the values for $E(X_t)$, $E(Y_t)$, $E(Z_t)$, $E(A_t)$ and $E(B_t)$. The time to reach steady state is $t \approx 260$ which is $\gg RT = 31$.

Case (II) Machine Interference Model

Discrete Case

Assume $C = 5$, $N = 20$, $\epsilon = 0.04$, $\mu = 0.1$, $m = 10$, $\epsilon_i = (N-i)\epsilon$, $\mu_i = i\mu$, for $0 \le i < c$, $\mu_i = c\mu$ for $c \le i \le N$ and $p_o(i) = 1/21$. Table 3 gives the probabilities $p_m(n)$ for $0 \le i \le 20$, the unconditional probabilities $Q_m(n)$ and the steady-state probabilities $p(n)$. The last five rows give the values for $E(X_m)$, $E(Y_m)$, $E(Z_m)$, $E(A_m)$ and $E(B_m)$. The epoch to reach steady state is $m \approx 220$ which is $\gg RT = 22$.

Continuous Case

Assume $t=10$ and rest of the parameters as in Table 3, Table 4 gives the probcbilities $p_t(n)$ for different $i(0 \le i \le 20)$, the unconditional probabilities $Q_t(n)$ and the steady-state probabilities $p(n)$. The time to reach steady

state is $t \approx 130$ which is $\gg RT = 20$.

4.8 CONCLUSION :

We have discussed a discrete-time Markovian Model Geom(n)/Geom(n)/r/N for a balking and reneging problem and obtained its transient solution. Numerical computations have been carried out for balking and reneging problem and also for a machine interference problem which is a particular case of balking and reneging problem. Computations have also been carried out for their counterpart in continuous time. Discrete time models are very important for areas such as Computer Science. An analogy has also been established between discrete time and continuous time models. Finally the accuracy of the eigen values (roots) that is needed in such problems is very very large because of the recurrence relations involved in computing the probabilities.

Table 4.11: probabilities $p_m(n)$, $Q_m(n)$, $p(n)$ and some important performance measures for Geom(n)/Geom(n)/c/N balkink and reneging problem with $c=5$, $N=20$, $m=10$, $\epsilon=0.5$, $\mu=0.05$, $\Omega=0.005$, $i=0,1,\ldots,20$ and $p_o(i) = 1/21$.

	0	1	2		18	19	20	$Q_m(n)$	$p(n)$
0	0.0060	0.0025	0.0010	...	0.0000	0.0000	0.0000	0.0005	0.0000
1	0.0480	0.0252	0.0124	...	0.0000	0.0000	0.0000	0.0045	0.0002
2	0.1552	0.1019	0.0620	...	0.0000	0.0000	0.0000	0.0183	0.0015
3	0.2673	0.2188	0.1643	...	0.0000	0.0000	0.0000	0.0432	0.0068
4	0.2712	0.2780	0.2574	...	0.0000	0.0000	0.0000	0.0674	0.0205
5	0.1673	0.2161	0.2468	...	0.0000	0.0000	0.0000	0.0754	0.0420
6	0.0654	0.1096	0.1580	...	0.0000	0.0000	0.0000	0.0721	0.0712
7	0.0167	0.0379	0.0711	...	0.0000	0.0000	0.0000	0.0679	0.1058
8	0.0026	0.0086	0.0219	...	0.0000	0.0000	0.0000	0.0654	0.1372
9	0.0003	0.0012	0.0045	...	0.0000	0.0000	0.0000	0.0645	0.1546
10	0.0000	0.0002	0.0006	...	0.0003	0.0000	0.0000	0.0642	0.1504
11	0.0000	0.0000	0.0000	...	0.0029	0.0005	0.0001	0.0642	0.1257
12	0.0000	0.0000	0.0000	...	0.0152	0.0040	0.0007	0.0641	0.0893
13	0.0000	0.0000	0.0000	...	0.0548	0.0203	0.0055	0.0631	0.0534
14	0.0000	0.0000	0.0000	...	0.1354	0.0690	0.0267	0.0637	0.0264
15	0.0000	0.0000	0.0000	...	0.2301	0.1603	0.0860	0.0619	0.0108
16	0.0000	0.0000	0.0000	...	0.2633	0.2525	0.1873	0.0561	0.0034
17	0.0000	0.0000	0.0000	...	0.1939	0.2619	0.2725	0.0439	0.0008
18	0.0000	0.0000	0.0000	...	0.0843	0.1673	0.2533	0.0263	0.0001
19	0.0000	0.0000	0.0000	...	0.0184	0.0570	0.1359	0.0104	0.0000
20	0.0000	0.0000	0.0000	...	0.0014	0.0072	0.0320	0.0019	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	3.6143	4.0819	4.5603	...	15.6806	16.4180	17.1554	9.9533	9.2255
E(Ym)	0.1077	0.2166	0.3869	...	10.6806	11.4180	12.1554	5.1826	4.2647
E(Zm)	0.2523	0.3736	0.5029	...	1.0000	1.0000	1.0000	0.8660	0.9711
E(Am)	0.1807	0.2041	0.2280	...	0.7840	0.8209	0.8578	0.4977	0.4613
E(Bm)	0.0584	0.1072	0.1727	...	0.2160	0.1791	0.1422	0.3339	0.4836

Table 4.2 : probabilities $p_t(n)$, $Q_t(n)$, $p(n)$ for m/m/c/n queue with t=10 and rest of the parameters as in table (4.1).

	0	1	2		18	19	20	$Q_m(n)$	p(n)
0	0.0209	0.0089	0.0038	...	0.0000	0.0000	0.0000	0.0017	0.0000
1	0.0891	0.0526	0.0286	...	0.0000	0.0000	0.0000	0.0095	0.0010
2	0.1807	0.1360	0.0932	...	0.0000	0.0000	0.0000	0.0256	0.0048
3	0.2314	0.2114	0.1767	...	0.0000	0.0000	0.0000	0.0463	0.0145
4	0.2087	0.2243	0.2219	...	0.0000	0.0000	0.0000	0.0625	0.0307
5	0.1391	0.1711	0.1954	...	0.0000	0.0000	0.0000	0.0667	0.0492
6	0.0763	0.1064	0.1390	...	0.0000	0.0000	0.0000	0.0673	0.0723
7	0.0349	0.0546	0.0808	...	0.0000	0.0000	0.0000	0.0664	0.0974
8	0.0133	0.0232	0.0152	...	0.0002	0.0001	0.0000	0.0654	0.1194
9	0.0043	0.0083	0.0050	...	0.0008	0.0003	0.0001	0.0647	0.1327
10	0.0012	0.0025	0.0014	...	0.0029	0.0011	0.0004	0.0644	0.1327
11	0.0003	0.0006	0.0003	...	0.0095	0.0038	0.0004	0.0643	0.1186
12	0.0000	0.0001	0.0001	...	0.0265	0.0120	0.0049	0.0642	0.0935
13	0.0000	0.0000	0.0000	...	0.0629	0.0325	0.0152	0.0638	0.0645
14	0.0000	0.0000	0.0000	...	0.1235	0.0745	0.0396	0.0628	0.0383
15	0.0000	0.0000	0.0000	...	0.1948	0.1409	0.0879	0.0601	0.0192
16	0.0000	0.0000	0.0000	...	0.2353	0.2119	0.1594	0.0542	0.0078
17	0.0000	0.0000	0.0000	...	0.2023	0.2399	0.2278	0.0438	0.0026
18	0.0000	0.0000	0.0000	...	0.1092	0.1868	0.2403	0.0291	0.0006
19	0.0000	0.0000	0.0000	...	0.0292	0.0834	0.1663	0.0139	0.0001
20	0.0000	0.0000	0.0000	...	0.0029	0.0128	0.0567	0.0035	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	3.5369	4.0253	4.5256	...	15.7106	16.4514	17.1922	9.9568	9.2848
E(Ym)	0.2110	0.3352	0.5137	...	10.7106	11.4514	12.1922	5.2352	4.3635
E(Zm)	0.2694	0.3668	0.4759	...	1.0000	1.0000	1.0000	0.8544	0.9488
E(Am)	0.1768	0.2013	0.2263	...	0.7855	0.8226	0.8596	0.4978	0.4642
E(Bm)	0.0872	0.1300	0.1847	...	0.2145	0.1774	0.1404	0.3291	0.4566

Table 4.3: probabilities $p_m(n)$, $Q_m(n)$, $p(n)$ and some important performance measures for Geom(n)/Geom(n)/c/N machine interference problem with c=5, N=20, m=10, ϵ=0.4, μ=0.1, i=0,1,....20 and $p_o(i)$ = 1/21.

	0	1	2		18	19	20	$Q_m(n)$	p(n)
0	0.0003	0.0002	0.0001	...	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0065	0.0044	0.0029	...	0.0000	0.0000	0.0000	0.0008	0.0000
2	0.0478	0.0354	0.0255	...	0.0000	0.0000	0.0000	0.0073	0.0027
3	0.1651	0.1346	0.1073	...	0.0000	0.0000	0.0000	0.0305	0.0161
4	0.2953	0.2683	0.2376	...	0.0000	0.0000	0.0000	0.0708	0.0532
5	0.2789	0.2861	0.2837	...	0.0000	0.0000	0.0000	0.0956	0.1021
6	0.1448	0.1737	0.1988	...	0.0000	0.0000	0.0000	0.0903	0.1392
7	0.0493	0.0727	0.0993	...	0.0000	0.0000	0.0000	0.0807	0.1624
8	0.0106	0.0205	0.0350	...	0.0000	0.0000	0.0000	0.0742	0.1624
9	0.0013	0.0037	0.0084	...	0.0008	0.0001	0.0000	0.0715	0.1392
10	0.0001	0.0004	0.0013	...	0.0059	0.0012	0.0001	0.0707	0.1021
11	0.0000	0.0000	0.0001	...	0.0269	0.0089	0.0019	0.0705	0.0638
12	0.0000	0.0000	0.0000	...	0.0796	0.0374	0.0130	0.0703	0.0239
13	0.0000	0.0000	0.0000	...	0.1605	0.1022	0.0511	0.0694	0.0150
14	0.0000	0.0000	0.0000	...	0.2279	0.1896	0.1291	0.0656	0.0055
15	0.0000	0.0000	0.0000	...	0.2293	0.2442	0.2191	0.0563	0.0019
16	0.0000	0.0000	0.0000	...	0.1620	0.2183	0.2537	0.0408	0.0004
17	0.0000	0.0000	0.0000	...	0.0782	0.1328	0.1983	0.0229	0.0001
18	0.0000	0.0000	0.0000	...	0.0243	0.0522	0.1002	0.0092	0.0000
19	0.0000	0.0000	0.0000	...	0.0043	0.0119	0.0296	0.0023	0.0000
20	0.0000	0.0000	0.0000	...	0.0003	0.0012	0.0039	0.0003	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	4.4846	4.7349	5.0022	...	14.4807	15.1456	15.8104	9.4599	7.7358
E(Ym)	0.2809	0.3972	0.5432	...	9.4807	10.1456	10.8104	4.6170	2.8301
E(Zm)	0.4851	0.5571	0.6266	...	1.0000	1.0000	1.0000	0.8906	0.9278
E(Am)	0.2242	0.2367	0.2501	...	0.7240	0.7573	0.7905	0.4730	0.3868
E(Bm)	0.1406	0.1833	0.2300	...	0.2760	0.2427	0.2095	0.3654	0.4770

Table 4.4: probabilities $p_t(n)$, $Q_t(n)$, $p(n)$ and some important performance measures for a m/m/c/N machine interference with t=10 amd rest of the paramemeters as in table 4.3.

	0	1	2		18	19	20	$Q_m(n)$	p(n)
0	0.0078	0.0054	0.0037	...	0.0000	0.0000	0.0000	0.0011	0.0006
1	0.0429	0.0325	0.0243	...	0.0000	0.0000	0.0000	0.0073	0.0044
2	0.1115	0.0923	0.0750	...	0.0000	0.0000	0.0000	0.0228	0.0167
3	0.1820	0.1634	0.1434	...	0.0000	0.0000	0.0000	0.0456	0.0402
4	0.2080	0.2008	0.1892	...	0.0000	0.0000	0.0000	0.0657	0.0683
5	0.1736	0.1793	0.1805	...	0.0002	0.0001	0.0000	0.0720	0.0874
6	0.1247	0.1381	0.1493	...	0.0005	0.0002	0.0001	0.0746	0.1049
7	0.0774	0.0920	0.1071	...	0.0015	0.0006	0.0003	0.0747	0.1175
8	0.0415	0.0530	0.0664	...	0.0040	0.0019	0.0008	0.0737	0.1222
9	0.0193	0.0262	0.0356	...	0.0099	0.0050	0.0024	0.0725	0.1173
10	0.0077	0.0112	0.0163	...	0.0225	0.0124	0.0063	0.0715	0.1032
11	0.0026	0.0041	0.0064	...	0.0457	0.0275	0.0153	0.0706	0.0826
12	0.0008	0.0013	0.0021	...	0.0820	0.0546	0.0335	0.0693	0.0594
13	0.0002	0.0003	0.0006	...	0.1281	0.0954	0.0650	0.0669	0.0380
14	0.0000	0.0001	0.0001	...	0.1706	0.1441	0.1104	0.0624	0.0213
15	0.0000	0.0000	0.0000	...	0.1887	0.1837	0.1607	0.0544	0.0102
16	0.0000	0.0000	0.0000	...	0.1668	0.1912	0.1950	0.0430	0.0041
17	0.0000	0.0000	0.0000	...	0.1115	0.1548	0.1893	0.0291	0.0013
18	0.0000	0.0000	0.0000	...	0.0518	0.0902	0.1378	0.0157	0.0003
19	0.0000	0.0000	0.0000	...	0.0144	0.0329	0.0669	0.0059	0.0001
20	0.0000	0.0000	0.0000	...	0.0018	0.0054	0.0162	0.0012	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	4.4244	4.7178	5.0294	...	14.5384	15.2087	15.8790	9.5082	8.0480
E(Ym)	0.5419	0.6793	0.8460	...	9.5384	10.2087	10.8790	4.7681	3.2672
E(Zm)	0.4477	0.5056	0.5644	...	0.9999	1.0000	1.0000	0.8575	0.8698
E(Am)	0.2212	0.2359	0.2515	...	0.7269	0.7604	0.7940	0.4754	0.4024
E(Bm)	0.1784	0.2108	0.2456	...	0.2729	0.2395	0.2060	0.3507	0.4235

CHAPTER 5

ESTIMATION OF PARAMETERS OF JACKSON NETWORKS WITH THREE NODES

5.1 INTRODUCTION :

In this chapter we have try to estimate the parameters involved in Jacksons Networks. As we know Jackson networks have been extended in several ways. First Jackson (1963) for open networks allowed state dependent exogenous arrival processes and state dependent internal service. The parameters of the exogenous poisson process depend upon the total number of customers present at that node. We consider a network of three service facilities customers can arrive from outside to any node according to poisson law. All servers at different node work according to exponential distribution when a customer complete a service at a particular node he goes to next node with some probability.

To estimate the parameters we have to use method maximum likelihood function, which require the joint probability density function for the number of customers at each node. This likelihood function of the parameters involved in the Jackson networks.

5.2 NOTATIONS :

The following parameters are involved in the networks with three nodes :

γ_i ≡ mean arrival rate at node i (i=1,2,3) follows according to poisson process.

μ_i ≡ mean service rate at node i (i=1,2,3) follows according to exponential distributions.

λ_i ≡ Total mean flow rate into node i (i=1,2,3).

$\lambda_i = \gamma_i + r_{1i}\ \lambda_1 + r_{2i}\ \lambda_2 + r_{3i}\ \lambda_3$.

$\rho_i = \lambda_i/\mu_i$.

r_{ij} ≡ probability that a customer complete service at node i (i=1,2,3) and he goes to next node j (j=1,2,3).

r_{i0} ≡ probability that a customer will leaves the network at node i upon completion of service.

Further we assume that there is no limit to the capacity at any node i (i=1,2,3).

5.3 ANALYSIS AND ESTIMATION :

Let us denote N_1,N_2 and N_3 be the random variables for the number of customers at node 1, node 2 and node 3 respectively. The joint probability density function of N_1,N_2 and N_3 is given by

$$L = P(N_1=n_1, N_2=n_2, N_3=n_3)$$

$$L = (1-\rho_1)\rho_1^{n_1} \; (1-\rho_2)\rho_2^{n_2} \; (1-\rho_3)\rho_3^{n_3} \longrightarrow (5.1)$$

Taking logarithm both sides, we have

$$\log L = \log(1-\rho_1) + \log(1-\rho_2) + \log(1-\rho_3)$$
$$+ n_1 \log\rho_1 + n_2 \log\rho_2 + n_3 \log\rho_3$$

On differentiating this equation with respect to ρ_1, ρ_2 and ρ_3, we get

$$\frac{\delta(\log L)}{\delta\rho_1} = \frac{-1}{(1-\rho_1)} + \frac{n_1}{\rho_1} = 0 \longrightarrow (5.2)$$

$$\frac{\delta(\log L)}{\delta\rho_2} = \frac{-1}{(1-\rho_2)} + \frac{n_2}{\rho_2} = 0 \longrightarrow (5.3)$$

$$\frac{\delta(\log L)}{\delta\rho_3} = \frac{-1}{(1-\rho_3)} + \frac{n_3}{\rho_3} = 0 \longrightarrow (5.4)$$

On solving the above likelihood equations, we can get the estimate of ρ_1, ρ_2 and ρ_3 as follows :

$$\left.\begin{aligned} \hat{\rho}_1 &= \frac{n_1}{(n_1+1)} \\ \hat{\rho}_2 &= \frac{n_2}{(n_2+1)} \\ \hat{\rho}_3 &= \frac{n_3}{(n_3+1)} \end{aligned}\right] \longrightarrow (5.5)$$

Alternatively, we can substitute the value of $\rho_i = \lambda_i/\mu_i$ $(i=1,2,3)$ in the joint probability density function given in (5.1), behave the likelihood function as

$$L = \left[1-\frac{\lambda_1}{\mu_1}\right]\left[1-\frac{\lambda_2}{\mu_2}\right]\left[1-\frac{\lambda_3}{\mu_3}\right]\left[\frac{\lambda_1}{\mu_1}\right]^{n_1}\left[\frac{\lambda_2}{\mu_2}\right]^{n_2}\left[\frac{\lambda_3}{\mu_3}\right]^{n_3} \longrightarrow (5.3)$$

On taking the logarithm both sides and differentially with respect to unknown parameters $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2$ and μ_3, we get the same estimated as obtain in equation (5.5).

Further, on using the following relation

$$\lambda_i = \gamma_i + \sum_{j=1}^{3} r_{ji}\, \lambda_j \longrightarrow (5.7)$$

i.e. when i=1

$$\lambda_1 = \gamma_1 + r_{11}\lambda_1 + r_{21}\lambda_2 + r_{31}\lambda_3$$

$$\left[1-r_{11}\right]\lambda_1 - r_{21}\lambda_2 - r_{31}\lambda_3 = \gamma_1 \longrightarrow (5.8)$$

Similarly, we can find out

$$-r_{12}\lambda_1 + \left[1-r_{22}\right]\lambda_2 - r_{32}\lambda_3 = \gamma_2 \longrightarrow (5.9)$$

$$-r_{13}\lambda_1 - r_{23}\lambda_2 + \left[1-r_{33}\right]\lambda_3 = \gamma_3 \longrightarrow (5.10)$$

Once we know the estimates of the unknown parameters λ_1, λ_2 and λ_3, we can obtain the estimates of γ_1, γ_2 and γ_3 using equation (5.8) to (5.10) only when routing probabilities r_{ij} (i=1,2,3 ; j=1,2,3) are known

$$\left.\begin{aligned}
\hat{\gamma}_1 &= \left(1-r_{11}\right)\hat{\lambda}_1 - r_{21}\hat{\lambda}_2 - r_{31}\hat{\lambda}_3 \\
\hat{\gamma}_2 &= -r_{12}\hat{\lambda}_1 + \left(1-r_{22}\right)\hat{\lambda}_2 - r_{32}\hat{\lambda}_3 \\
\hat{\gamma}_3 &= -r_{13}\hat{\lambda}_1 - r_{23}\hat{\lambda}_2 + \left(1-r_{33}\right)\hat{\lambda}_3
\end{aligned}\right] \longrightarrow (5.11)$$

Similar estimation can be done on closed jackson networks which are particular case of Open Jackson Networks. For closed jackson networks, we have

$$\gamma_i = 0 \text{ and } \gamma_{i0} = 0 \qquad (i=1,2,3)$$

Then on solving equation (5.8) to (5.10), we can obtain the estimates of λ_1, λ_2 and λ_3 respectively. We can write these equations in matrix as follows

$$AX = 0$$

$$\text{where } A = \begin{bmatrix} (1-r_{11}) & -r_{21} & -r_{31} \\ -r_{12} & -(1-r_{22}) & -r_{32} \\ -r_{13} & -r_{23} & -(1-r_{33}) \end{bmatrix} \quad \text{and } X = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

If matrix A is a non singular matrix then all estimates of λ_1, λ_2 and λ_3 becomes zero which is not possible. Hence, A must be a singular matrix.

5.4 CONCLUSION :

In the estimation of the parameters of Jackson Networks we assume that the number of servers is equal to

number of customers in the systems which may not be true in general. Further, this can be extended to class of networks which allow for different class of customers at different nodes. This will be complicated to estimate of the parameters. The main purpose of estimating the parameters is to know the behavior of the system.

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